Locality for Classical Logic

Kai Brünnler
Institut für angewandte Mathematik und Informatik
Neubrückstr. 10, CH – 3012 Bern, Switzerland

Abstract In this paper we will see deductive systems for classical propositional and predicate logic in the calculus of structures. Like sequent systems, they have a cut rule which is admissible. Unlike sequent systems, they drop the restriction that rules only apply to the main connective of a formula: their rules apply anywhere *deeply* inside a formula. This allows to observe very clearly the symmetry between identity axiom and the cut rule. This symmetry allows to reduce the cut rule to atomic form in a way which is dual to reducing the identity axiom to atomic form. We also reduce weakening and even contraction to atomic form. This leads to inference rules that are *local*: they do not require the inspection of expressions of arbitrary size.

Keywords cut elimination, deep inference, locality

Mathematics Subject Classification 03F05 Cut-elimination and normal-form theorems, 03F07 Structure of proofs

Table of Contents

1 Introduction

The design of the logical rules in Gentzen's sequent system LK [11] allows to inductively replace instances of the identity axiom on compound formulas by instances on smaller formulas. The identity axiom can thus be reduced to atomic form, i.e.

$$\overline{A \vdash A} \qquad \text{can be equivalently replaced by} \qquad \overline{a \vdash a} \quad ,$$

where a is an atom. This property is desirable: in general it is desirable to build complex objects from primitives that are as simple as possible and the atomic version of the rule is simpler than the general version. Indeed, this property simplifies somewhat the frequent case analysis of what happens to a formula during the course of going up in a proof.

A natural question is thus whether other rules in LK are similarly reducible to atomic form, but it is not difficult to see that this is not the case. The cut is of course reducible to atomic form, and trivially so once we have established cut elimination. But this requires a complex argument which is nowhere nearly as simple as the reduction of the identity axiom. The observation that contraction cannot be reduced to atomic form can be found in [6].

It turns out that it is possible to reduce identity axiom, cut, weakening and contraction to atomic form once we leave the sequent calculus and use the *calculus of structures* [13], a formalism which can be seen as a generalisation of the sequent calculus. Inference rules in the sequent calculus only apply at the main connective of a formula. This restriction was lifted already in Schütte's calculus of positive and negative parts [26], which allows inference rules to apply at certain places inside a formula. The calculus of structures can be seen as taking Schütte's idea to the ultimate: inference rules apply *anywhere* deep inside a formula, just like rules in term rewriting [1].

Thanks to deep inference our systems have several features that sequent systems do not have. We can clearly observe the duality between the identity axiom and the cut rule which take the following form:

$$identity\,\frac{true}{A \vee \bar{A}} \qquad \text{and} \qquad cut\,\frac{A \wedge \bar{A}}{false} \quad .$$

One can be obtained from the other by exchanging premise and conclusion and negating them. We see that this is the notion known under the name *contrapositive*.

Thanks to this symmetry, the cut is reducible to atomic form in the same way that the identity axiom is reducible to atomic form – cut elimination is not needed for that.

Contraction is decomposed into two rules: atomic contraction, which only applies to atoms and a rule baptised *medial* which is due to Tiu [8]. It corresponds to the inference

$$\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \quad ,$$

which also occurs in Blass' work on game semantics [2] and is thus sometimes referred to as Blass' principle. The decomposition of contraction inspired Lamarche and Straßburger to a new notion of classical proof net [20, 19].

Weakening can also be reduced to atomic form. Consequently all the rules that duplicate formulas, erase formulas or check the equality of formulas only need to duplicate, erase or check atoms. All rules only affect a small, bounded portion of the formula they are applied to, a property that I call *locality*.

All the rules in our systems are sound in the strong sense that the premise implies the conclusion. The $\mathsf{R}\forall$-rule in the sequent calculus is only sound in the weaker sense that the validity of the premise implies the validity of the conclusion. Our rule corresponds to the inference

$$\frac{\forall x(A \supset B)}{\forall xA \supset \forall xB} \quad ,$$

which happens to be exactly what is required in order to reduce identity axiom and cut to atomic form. Soundness in this stronger sense allows us to prove a deduction theorem which has no analogue in the one-sided sequent calculus. The rule above also allows us to restrict formulas that occur in proofs to sentences, i.e. formulas that do not contain free variables. Quine already found it desirable to avoid the use of free variables in proofs and did so by using the above rule as an axiom in his (Hilbert-style) system given in [23].

The calculus of structures was conceived by Guglielmi in order to express a logical system with a connective that resembles sequential composition in process algebras [13, 16, 17, 9]. The ideas developed in [13] have also been explored in the setting of classical logic: in [8] to obtain locality for propositional logic and in [4] to obtain a particularly simple cut elimination procedure for propositional logic, which does not require an induction on the cut rank. Both of these works are contained in my PhD thesis [5] which also treats predicate logic. The present work is a revised version of a part of this thesis. The cut elimination procedure from [4] has been extended to predicate logic in [7].

The calculus of structures has also been employed by Straßburger to give systems for linear logic which neither suffer a nondeterministic context-splitting in the tensor rule nor a global promotion rule [28, 29].

This paper is structured as follows: in the next section I introduce the basic notions of the proof-theoretic formalism used, the calculus of structures. Section 3 is devoted to classical propositional logic and Section 4 to predicate logic.

2 The Calculus of Structures

Definition 2.1. Propositional variables p and their negations $\bar{p}$ are *atoms*. Atoms are denoted by a, b, c and so on. The *formulas* of the language KS are generated by

$$S ::= \mathsf{f} \mid \mathsf{t} \mid a \mid [\, S, S \,] \mid (\, S, S \,) \quad ,$$

where f and t are the units *false* and *true*, $[S_1, S_2]$ is a *disjunction* and (S_1, S_2) is a *conjunction*. Formulas are denoted by S, P, Q, R, T, U and V. *Formula contexts*, denoted by $S\{\ \}$, are formulas with one occurrence of $\{\ \}$, the *empty* *context* or *hole*. $S\{R\}$ denotes the formula obtained by filling the hole in $S\{\ \}$ with R. We drop the curly braces when they are redundant: for example, $S[R,T]$ is short for $S\{[R,T]\}$. A formula R is a *subformula* of a formula T if there is a context $S\{\ \}$ such that $S\{R\}$ is T.

Definition 2.2. We define $\bar{S}$, the *negation* of the formula S, as follows:

$$\begin{array}{lll} \bar{\mathsf{f}} = \mathsf{t} & \overline{[R,T]} = (\bar{R},\bar{T}) & \\ \bar{\mathsf{t}} = \mathsf{f} & \overline{(R,T)} = [\bar{R},\bar{T}] & \bar{\bar{p}} = p \end{array}$$

Notation 2.3. We use $[R,T,U]$ to abbreviate a formula that could be either $[R,[T,U]]$ or $[[R,T],U]$, and likewise for an arbitrary number of formulas in a disjunction. We do the same for conjunction.

What we have defined above are just formulas in negation normal form. The sequent calculus has two types of objects to deduce over, namely formulas and sequents. The inference systems that we will see will have just one type of objects, namely formulas. Since formulas will have to play the role of sequents it turns out that the outfix notation for connectives is more convenient than the standard infix notation. For the same reason it will be convenient to equip connectives of formulas with the same properties that the comma in a sequent typically enjoys:

Definition 2.4. We define a *syntactic equivalence* on formulas which is the smallest congruence relation induced by commutativity and associativity of conjunction and disjunction as well as the following equations for the units:

$$\begin{array}{ll} [R,\mathsf{f}] = R & [\mathsf{t},\mathsf{t}] = \mathsf{t} \\ (R,\mathsf{t}) = R & (\mathsf{f},\mathsf{f}) = \mathsf{f} \end{array} \quad .$$

Definition 2.5. An *inference rule* is a triple (ρ, R, T), where R and T are formulas that may contain schematic formulas and schematic atoms. It is written

$$\rho \frac{S\{T\}}{S\{R\}} \quad ,$$

where ρ is the *name* of the rule, $S\{T\}$ is its *premise* and $S\{R\}$ is its *conclusion*. An *instance of an inference rule* consists of a context $S\{\ \}$ together with the inference rule in which all schematic formulas and schematic atoms are replaced by formulas and atoms, respectively. In an instance of an inference rule the formula taking the place of R is its *redex*, the formula taking the place of T is its *contractum* and the context taking the place of $S\{\ \}$ is its *context*. A (*deductive*) *system* $\mathcal{S}$ is a set of inference rules.

An inference rule is thus just a rewrite rule as known from term rewriting with the minor difference that there are two kinds of variables, one for atoms and one for arbitrary formulas, and the notational difference that the context is made explicit. For example, the rule ρ from the previous definition seen top-down corresponds to a rewrite rule $T \rightarrow R$.

We now define derivations which are top-down symmetric, contrary to the derivations in the sequent calculus, which are trees and thus asymmetric:

Definition 2.6. A *derivation* Δ in a certain deductive system is a finite sequence of instances of inference rules in the system:

$$\pi \frac{T}{V} \atop {\pi' \frac{}{\vdots} \atop {\rho' \frac{}{U} \atop \rho \frac{}{R}}} \quad .$$

A derivation can consist of just one formula. The topmost formula in a derivation is called the *premise* of the derivation, and the formula at the bottom is called its *conclusion*. The *size* of the derivation is the number of instances of inference rules, without counting the equivalence rule.

Sometimes in the literature the word derivation is used as being synonymous to the word proof. Note that here it instead corresponds to the more general notion of partial proof.

Definition 2.7. There is a special inference rule, the *equivalence rule*

$$= \frac{T}{R} \quad ,$$

where R and T are syntactically equivalent formulas. This rule is contained in every deductive system without being explicitly mentioned. Obvious instances of it are usually omitted from derivations. This means that, morally speaking, we are not deducing over formulas but over equivalence classes of formulas.

Notation 2.8. A derivation Δ whose premise is T, whose conclusion is R, and whose inference rules are in $\mathcal{S}$ is denoted by

$$\begin{array}{c} T \\ {\scriptstyle\Delta} \Vert \mathcal{S} \\ R \end{array} \quad .$$

Definition 2.9. Given a derivation Δ and a context $S\{\ \}$, the derivation $S\{\Delta\}$ is obtained by replacing each formula U in Δ by $S\{U\}$. Given two derivations Δ_1 from U to T and Δ_2 from T to R we define the derivation $\Delta_1; \Delta_2$ from U to R as the vertical composition of these two derivations in the obvious way. Given two derivations Δ_1 from R_1 to T_1 and Δ_2 from R_2 to T_2 we define the derivation (Δ_1, Δ_2) from (R_1, R_2) to (T_1, T_2) as $(R_1, \Delta_2); (\Delta_1, T_2)$ and we do likewise for $[\Delta_1, \Delta_2]$.

Definition 2.10. A rule ρ is *derivable* for a system $\mathcal{S}$ if for every instance of $\rho \frac{T}{R}$ there is a derivation $\begin{array}{c} T \\ \Vert \mathcal{S} \\ R \end{array}$.

The symmetry of derivations, where both premise and conclusion are arbitrary formulas, is broken in the notion of *proof*:

Definition 2.11. A *proof* is a derivation whose premise is the unit t. A proof Π of R in system $\mathcal{S}$ is denoted by

$$\Pi \, \begin{array}{c} \| \, \mathcal{S} \\ R \end{array} \quad .$$

3 Propositional Logic

In this section we see deductive systems for classical propositional logic with inference rules that apply deep inside formulas. Thanks to that, we observe a vertical symmetry that can not be observed in the sequent calculus.

The section is structured as follows: I first present system SKSg, a set of inference rules for classical propositional logic which is closed under a notion of duality. I then translate derivations of a one-sided sequent system into this system, and vice versa. This establishes soundness and completeness with respect to classical propositional logic as well as cut admissibility. In the following I obtain an equivalent system, named SKS, in which identity, cut, weakening and contraction are reduced to atomic form. This entails locality of the system.

3.1 A Deep Inference System

Taking a close look at the identity axiom and the cut rule in the sequent calculus [11], in its one-sided version [25, 31], we notice a certain duality:

$$\mathsf{Ax}\,\frac{}{\vdash A, \bar{A}} \qquad\qquad \mathsf{Cut}\,\frac{\vdash \Phi, A \quad \vdash \Psi, \bar{A}}{\vdash \Phi, \Psi} \quad .$$

When seen bottom-up, the cut introduces an arbitrary formula A together with its negation $\bar{A}$. The identity axiom also introduces an arbitrary formula A and its negation $\bar{A}$, but this time when seen top-down. Clearly, the two rules are intimately related. However, their duality is obscured by the fact that a certain top-down symmetry is inherently broken in the sequent calculus: derivations are trees, and trees are top-down asymmetric.

Since the calculus of structures abandons the tree-shape of derivations, we can reveal the duality between the two rules:

Definition 3.1. We define the following two inference rules where the rule i↓ is called *identity* and the rule i↑ is called *cut*:

$$\mathsf{i}{\downarrow}\,\frac{S\{\mathsf{t}\}}{S[R, \bar{R}]} \qquad\qquad \mathsf{i}{\uparrow}\,\frac{S(R, \bar{R})}{S\{\mathsf{f}\}} \quad .$$

The duality between the two is well-known under the name *contrapositive*:

Definition 3.2. The *dual* of an inference rule is obtained by exchanging premise and conclusion and replacing each connective by its De Morgan dual.

The rules i↓ and i↑ respectively indeed correspond to the identity axiom and the cut rule in the sequent calculus, as we will see shortly.

Definition 3.3. A system of inference rules is called *symmetric* if for each of its rules it also contains the dual rule.

A symmetric system for classical propositional logic is shown in Figure 1. Note that a symmetric system that contains the identity rule by definition contains the cut rule as well, so in general we can read "symmetric" as "contains cut". The name of the system is SKSg, where the first S stands for "symmetric", K stands for "klassisch" as in Gentzen's LK and the second S says that it is a system in the calculus of structures. Small letters are appended to the name of a system to denote variants. In this case, the g stands for "global" or "general", meaning that rules are not restricted to atoms: they can be applied to arbitrary formulas. We will see in the next section that this system is sound and complete for classical propositional logic.

$$\mathsf{i}{\downarrow}\,\frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \qquad \mathsf{i}{\uparrow}\,\frac{S(R,\bar{R})}{S\{\mathsf{f}\}}$$

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{w}{\downarrow}\,\frac{S\{\mathsf{f}\}}{S\{R\}} \qquad \mathsf{w}{\uparrow}\,\frac{S\{R\}}{S\{\mathsf{t}\}}$$

$$\mathsf{c}{\downarrow}\,\frac{S[R,R]}{S\{R\}} \qquad \mathsf{c}{\uparrow}\,\frac{S\{R\}}{S(R,R)}$$

Figure 1: System SKSg

The rules $\mathsf{s}, \mathsf{w}{\downarrow}$ and $\mathsf{c}{\downarrow}$ are called respectively *switch*, *weakening* and *contraction*. Their dual rules carry the same name prefixed with a "co-", so e.g. $\mathsf{w}{\uparrow}$ is called *co-weakening*. Rules $\mathsf{i}{\downarrow}, \mathsf{w}{\downarrow}, \mathsf{c}{\downarrow}$ are called *down-rules* and their duals are called *up-rules*. The dual of the switch rule is the switch rule itself: it is *self-dual*.

The notion of duality generalises from rules to derivations:

Definition 3.4. The *dual* of a derivation is obtained by turning it upside-down and replacing each rule, each connective and each atom by its dual. For example

$$\mathsf{c}{\downarrow}\,\dfrac{\mathsf{w}{\uparrow}\,\dfrac{[(a,\bar{b}),a]}{[a,a]}}{a} \qquad \text{is dual to} \qquad \mathsf{w}{\downarrow}\,\dfrac{\mathsf{c}{\uparrow}\,\dfrac{\bar{a}}{(\bar{a},\bar{a})}}{([\bar{a},b],\bar{a})} \quad .$$

This vertical symmetry (i.e. symmetry with respect to a horizontal axis), which is depicted in Figure 2, is very much the same as the horizontal left-right symmetry of proofs in the two-sided sequent calculus. The crucial difference is that it

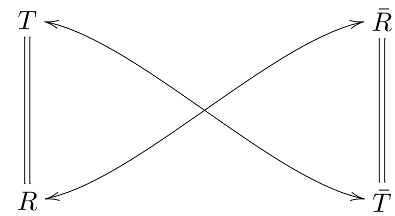


Figure 2: Two dual derivations

is in line with the duality of cut and identity while the symmetry of the sequent calculus is in some sense orthogonal to the duality of these two rules.

Note that the notion of proof is an asymmetric one: the dual of a proof is not a proof, it is a *refutation*.

3.2 Correspondence to the Sequent Calculus

The sequent system that is most similar to system SKSg is the one-sided system GS1p [31], also called *Gentzen-Schütte system* or *Tait-style system*. In this section we consider a version of GS1p with multiplicative context treatment and constants $\top$ and $\bot$, and we translate its derivations to derivations in SKSg and vice versa. Both translations increase the size of the derivation at most linearly. Translating from the sequent calculus to the calculus of structures is straightforward, in particular, no new cuts are introduced in the process. But to translate in the other direction we have to simulate deep inferences in the sequent calculus, which is done by using the cut rule.

One consequence of those translations is that system SKSg is sound and complete for classical propositional logic. Another consequence is cut elimination: one can translate a proof with cuts in SKSg to a proof in GS1p + Cut, apply cut elimination for GS1p, and translate back the resulting cut-free proof to obtain a cut-free proof in SKSg.

Definition 3.5. *Formulas* are denoted by A and B. They contain negation only on atoms and may contain the constants $\top$ and $\bot$. Multisets of formulas are denoted by Φ and Ψ. The empty multiset is denoted by $\emptyset$. In $A_1, \ldots, A_h$, where $h \geq 0$, a formula denotes the corresponding singleton multiset and the comma denotes multiset union. *Sequents*, denoted by Σ, are multisets of formulas. *Derivations* are defined as usual and denoted by Δ or $\begin{array}{c}\Sigma_1 \;\cdots\; \Sigma_h \\ \bigtriangledown\Delta \\ \Sigma\end{array}$, where $h \geq 0$, the sequents $\Sigma_1, \ldots, \Sigma_h$ are the *premises* and Σ is the *conclusion*. Proofs, denoted by Π, are derivations where each leaf is an instance of Ax or of $\top$. The *size* of a derivation is the number of instances of inference rules.

It is rather obvious how to translate from formulas of KS to formulas of GS1p and back, it is just a change between infix and outfix notation and between

$$
\top\,\frac{}{\vdash \top}
\qquad\qquad
\mathsf{Ax}\,\frac{}{\vdash A, \bar{A}}
$$

$$
\mathsf{R}\wedge\,\frac{\vdash \Phi, A \quad \vdash \Psi, B}{\vdash \Phi, \Psi, A \wedge B}
\qquad\qquad
\mathsf{R}\vee\,\frac{\vdash \Phi, A, B}{\vdash \Phi, A \vee B}
$$

$$
\mathsf{RC}\,\frac{\vdash \Phi, A, A}{\vdash \Phi, A}
\qquad\qquad
\mathsf{RW}\,\frac{\vdash \Phi}{\vdash \Phi, A}
$$

Figure 3: $\mathsf{GS1p}$: a one-sided sequent system for propositional logic

different symbols for the units. We translate a multiset (and thus a sequent) consisting of the formulas $A_1, \ldots, A_n$ into a disjunction $[A_1, \ldots, A_n]$ and the empty multiset into the unit f. In order to not clutter up notation too much, we just use the same letter, say Σ, to denote a sequent if it occurs inside a sequent calculus derivation and to denote the corresponding formula if it occurs in a derivation in the calculus of structures.

From the Sequent Calculus to the Calculus of Structures

Theorem 3.6. For every derivation $\begin{array}{c}\Sigma_1 \ \cdots \ \Sigma_h \\ \bigtriangledown \\ \Sigma\end{array}$ in $\mathsf{GS1p} + \mathsf{Cut}$ there exists a derivation $\begin{array}{c}(\Sigma_1, \ldots, \Sigma_h) \\ \Big\| \ \mathsf{SKSg} \setminus \{\mathsf{c}\uparrow, \mathsf{w}\uparrow\} \\ \Sigma\end{array}$ with the same number of cuts.

Proof. By structural induction on the given derivation Δ. If $\Delta = \Sigma$ then take Σ. If $\Delta = \top\,\frac{}{\vdash \top}$ then take t. If $\Delta = \mathsf{Ax}\,\frac{}{\vdash A, \bar{A}}$ then take $\mathsf{i}\downarrow\,\frac{\mathsf{t}}{[A, \bar{A}]}$.

In the case of the $\mathsf{R}\wedge$ rule, we have a derivation

$$
\Delta \;=\; \begin{array}{c}
\begin{array}{cc}\Sigma_1 \ \cdots \ \Sigma_k & \Sigma'_1 \ \cdots \ \Sigma'_l \\ \bigtriangledown & \bigtriangledown\end{array} \\
\mathsf{R}\wedge\,\dfrac{\vdash \Phi, A \qquad\qquad \vdash \Psi, B}{\vdash \Phi, \Psi, A \wedge B}
\end{array} \quad .
$$

By induction hypothesis we obtain derivations

$$\begin{array}{c}(\Sigma_1,\dots,\Sigma_k)\\ \Delta_1 \Big\Vert \, \mathsf{SKSg}\setminus\{\mathsf{c}\uparrow,\mathsf{w}\uparrow\}\\ [\Phi,A]\end{array} \quad \text{and} \quad \begin{array}{c}(\Sigma'_1,\dots,\Sigma'_l)\\ \Delta_2 \Big\Vert \, \mathsf{SKSg}\setminus\{\mathsf{c}\uparrow,\mathsf{w}\uparrow\}\\ [\Psi,B]\end{array} \quad .$$

The derivation in SKSg we are looking for is obtained by composing Δ_1 and Δ_2 and applying the switch rule twice:

$$\begin{array}{c}(\Sigma_1,\dots,\Sigma_k,\Sigma'_1,\dots,\Sigma'_l)\\ (\Delta_1,\Delta_2) \Big\Vert \, \mathsf{SKSg}\setminus\{\mathsf{c}\uparrow,\mathsf{w}\uparrow\}\\ \mathsf{s}\,\dfrac{([\Phi,A],[\Psi,B])}{\mathsf{s}\,\dfrac{[\Psi,([\Phi,A],B)]}{[\Phi,\Psi,(A,B)]}}\end{array} \quad .$$

The other cases are similar, where

$$\mathsf{Cut}\,\frac{\vdash \Phi,A \quad \vdash \Psi,\bar{A}}{\vdash \Phi,\Psi} \quad \text{translates to} \quad \mathsf{s}\,\dfrac{([\Phi,A],[\Psi,\bar{A}])}{\mathsf{s}\,\dfrac{[\Phi,(A,[\Psi,\bar{A}])]}{\mathsf{i}\uparrow\,\dfrac{[\Phi,\Psi,(A,\bar{A})]}{=\,\dfrac{[\Phi,\Psi,\mathsf{f}]}{[\Phi,\Psi]}}}} \quad ,$$

$$\mathsf{RC}\,\frac{\vdash \Phi,A,A}{\vdash \Phi,A} \quad \text{translates to} \quad \mathsf{c}\downarrow\,\frac{[\Phi,A,A]}{[\Phi,A]} \quad ,$$

$$\mathsf{RW}\,\frac{\vdash \Phi}{\vdash \Phi,A} \quad \text{translates to} \quad =\,\dfrac{\Phi}{\mathsf{w}\downarrow\,\dfrac{[\Phi,\mathsf{f}]}{[\Phi,A]}} \quad .$$

□

Clearly, the size of the resulting derivation in the calculus of structures is roughly the same as the size of the original derivation in the sequent calculus. In the worst case, in which the original derivation consists entirely of cuts, the size is increased by a factor of three.

Corollary 3.7.

1. If a sequent Σ has a proof in $\mathsf{GS1p}$ then Σ has a proof in $\mathsf{SKSg}\setminus\{\mathsf{i}\uparrow,\mathsf{c}\uparrow,\mathsf{w}\uparrow\}$.

2. If a sequent Σ has a proof in $\mathsf{GS1p}+\mathsf{Cut}$ then Σ has a proof in $\mathsf{SKSg}\setminus\{\mathsf{c}\uparrow,\mathsf{w}\uparrow\}$.

Lemma 3.8. For every two formulas A, B and every formula context $C\{\ \}$ there exists a derivation

$$\begin{array}{c} \vdash A, \bar{B} \\ \bigtriangledown \\ \vdash C\{A\}, \overline{C\{B\}} \end{array}$$

in $\mathsf{GS1p}$.

Theorem 3.9. For every derivation $\begin{array}{c} Q \\ \Big\Vert \mathsf{SKSg} \\ P \end{array}$ there exists a derivation $\begin{array}{c} \vdash Q \\ \bigtriangledown \\ \vdash P \end{array}$ in $\mathsf{GS1p} + \mathsf{Cut}$.

Proof. We construct the sequent derivation by induction on the length of the given derivation Δ in SKSg. If Δ consists of just one formula P, then P and Q are the same. Take $\vdash P$. If length of Δ is greater than zero we single out the topmost rule instance in Δ:

$$\begin{array}{c} Q \\ \Delta \Big\Vert \mathsf{SKSg} \\ P \end{array} \quad = \quad \begin{array}{c} \rho \dfrac{S\{T\}}{S\{R\}} \\ \Delta' \Big\Vert \mathsf{SKSg} \\ P \end{array}$$

The corresponding derivation in $\mathsf{GS1p}$ will be as follows:

$$\begin{array}{c} \overset{\Pi}{\bigtriangledown} \\ \vdash R, \bar{T} \\ \overset{\Delta_1}{\bigtriangledown} \\ \mathsf{Cut}\,\dfrac{\vdash S\{R\}, \overline{S\{T\}} \qquad \vdash S\{T\}}{\vdash S\{R\}} \\ \overset{\Delta_2}{\bigtriangledown} \\ \vdash P \end{array} \quad ,$$

where Δ_1 exists by Lemma 3.8 and Δ_2 exists by induction hypothesis. The proof Π depends on the rule ρ. It is easy to check that the proof Π exists for all the rules of SKSg, let us see just the case of the switch rule,

$$\mathsf{s}\,\frac{S([U, V], T)}{S[(U, T), V]} \quad ,$$

for which we have

$$\mathsf{R}\vee^2\,\dfrac{\mathsf{R}\wedge\,\dfrac{\mathsf{R}\wedge\,\dfrac{\mathsf{Ax}\,\dfrac{}{\vdash U,\bar{U}}\qquad \mathsf{Ax}\,\dfrac{}{\vdash V,\bar{V}}}{\vdash U,\bar{U}\wedge\bar{V},V}\qquad \mathsf{Ax}\,\dfrac{}{\vdash T,\bar{T}}}{\vdash (U\wedge T),V,(\bar{U}\wedge\bar{V}),\bar{T}}}{\vdash (U\wedge T)\vee V,(\bar{U}\wedge\bar{V})\vee\bar{T}}\quad .$$

□

When checking all the cases in the proof above, we see that the size of the derivation grows at most by a factor of $8+3\cdot d$, where d is the maximum depth in which rules are applied in the original derivation. It is important, however, to keep in mind that we introduce cuts.

Corollary 3.10. If a formula S has a proof in SKSg then $\vdash S$ has a proof in $\mathsf{GS1p}+\mathsf{Cut}$.

3.3 Soundness, Completeness and Cut Admissibility

Soundness and completeness of SKSg, i.e. the fact that a formula has a proof if and only if it is valid, follows from soundness and completeness of $\mathsf{GS1p}$ by Corollaries 3.7 and 3.10. Moreover, a formula T implies a formula R if and only if there is a derivation from T to R, which follows from soundness and completeness and the following theorem:

Theorem 3.11 (Deduction Theorem)**.**

There is a derivation $\begin{array}{c} T \\ \Vert\,\mathsf{SKSg} \\ R \end{array}$ if and only if there is a proof $\begin{array}{c} \top\!\!\top \\ \Vert\,\mathsf{SKSg} \\ [\bar{T},R] \end{array}$.

Proof. A proof Π can be obtained from a given derivation Δ as follows:

$$\begin{array}{c} T \\ \Delta\,\Vert\,\mathsf{SKSg} \\ R \end{array} \quad\rightsquigarrow\quad \begin{array}{c} \mathsf{i}\!\downarrow\dfrac{\mathsf{t}}{[\bar{T},T]} \\ {\scriptstyle [\bar{T},\Delta]}\,\Vert\,\mathsf{SKSg} \\ [\bar{T},R] \end{array} \quad ,$$

and a derivation Δ from a given proof Π as follows:

$$\begin{array}{c} \top\!\!\top \\ \Pi\,\Vert\,\mathsf{SKSg} \\ [\bar{T},R] \end{array} \quad\rightsquigarrow\quad \begin{array}{c} T \\ {\scriptstyle (T,\Pi)}\,\Vert\,\mathsf{SKSg} \\ \mathsf{i}\!\uparrow\dfrac{\mathsf{s}\,\dfrac{(T,[\bar{T},R])}{[R,(T,\bar{T})]}}{R} \end{array} \quad .$$

□

If one is just interested in the provability of a formula, i.e. the derivability of a formula from the premise t as opposed to the derivability of a formula from some arbitrary given premise, then the up-rules of system SKSg, i.e. $\mathsf{i}\uparrow$, $\mathsf{w}\uparrow$ and $\mathsf{c}\uparrow$, are superfluous. By removing them we obtain the system

$$\mathsf{KSg} = \{\mathsf{i}\downarrow, \mathsf{s}, \mathsf{w}\downarrow, \mathsf{c}\downarrow\} \quad .$$

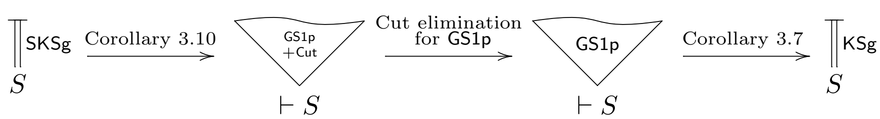
Definition 3.12. A rule ρ is *admissible* for a system $\mathcal{S}$ if for every proof $\begin{array}{c}\overline{\|}\,\mathcal{S}\cup\{\rho\}\\ S\end{array}$ there is a proof $\begin{array}{c}\overline{\|}\,\mathcal{S}\\ S\end{array}$. Two systems $\mathcal{S}$ and $\mathcal{S}'$ are (*weakly*) *equivalent* if for every proof $\begin{array}{c}\overline{\|}\,\mathcal{S}\\ R\end{array}$ there is a proof $\begin{array}{c}\overline{\|}\,\mathcal{S}'\\ R\end{array}$, and vice versa. Two systems $\mathcal{S}$ and $\mathcal{S}'$ are *strongly equivalent* if for every derivation $\begin{array}{c}T\\ \|\,\mathcal{S}\\ R\end{array}$ there is a derivation $\begin{array}{c}T\\ \|\,\mathcal{S}'\\ R\end{array}$, and vice versa.

The admissibility of all the up-rules for system KSg follows from cut admissibility in $\mathsf{GS1p}$ and the translation from the previous section:

Theorem 3.13 (Cut Admissibility)**.**

1. The rules $\mathsf{i}\uparrow$, $\mathsf{w}\uparrow$ and $\mathsf{c}\uparrow$ are admissible for system KSg.
2. The systems SKSg and KSg are equivalent.

Proof.



□

This theorem can also be proved without relying on the sequent calculus, see [4, 7].

The systems SKSg and KSg are not strongly equivalent. The cut rule, for example, can clearly not be derived in system KSg since there is no way of introducing new atoms going up. So, when a formula R implies a formula T then there is not necessarily a derivation from R to T in KSg, while there is one in SKSg. While the asymmetric, cut-free system is useful for proving formulas, we therefore have to use the symmetric system (i.e. the system with cut) for deriving conclusions from premises.

As a result of cut elimination, sequent systems fulfill the subformula property. Our systems do not distinguish between formulas and sequents, so technically of course they do not fulfill the subformula property – just as sequent systems do not fulfill a "subsequent property". However, seen bottom-up, in system KSg no rule introduces new atoms. It thus satisfies one main aspect of the subformula property: when given a conclusion of a rule there is only a finite number of premises to choose from.

3.4 Atomicity and Locality

Consider the contraction rule in the sequent calculus :

$$\frac{\vdash \Phi, A, A}{\vdash \Phi, A} \quad .$$

Here, going from bottom to top in constructing a proof, a formula A of arbitrary size is duplicated. Whatever mechanism performs this duplication, it has to inspect all of A, so it has to have a *global* view on A. Having a *local* view on a bounded portion of A is not enough.

I see two reasons why such a global behaviour is undesirable.

First, say that we want to measure the computational effort required for proof-checking. The effort required for checking the correctness of a given instance of the contraction rule depends on the size of the formula that is duplicated. The usual measures on proofs, like the depth or the number of instances of inference rules thus are not suitable for the complexity of proof-checking. A good measure would be more complicated, as it would have to look inside the rule instances.

Second, say that we want to implement contraction on a distributed system, where each processor has a limited amount of local memory. The formula A could be spread over a number of processors. In that case, no single processor has a global view on it.

I should stress that, given a suitable implementation, both of these objections become irrelevant. It is certainly possible to represent sequents in such a way that the contraction rule can be proof-checked in constant time just as it is possible let several processors duplicate a formula which is distributed among them. However, all the problems of a proof-theoretic system that are solved in its implementation of course widen the gap between the original system and its implementation. It may thus be worthwile to solve these problems already *inside* the proof-theoretic system, i.e. by avoiding global rules. This is what we set out to do in this section. We achieve locality by reducing the problematic rules to their atomic forms.

To reduce contraction we need to add the *medial* rule [8]:

$$\mathsf{m}\,\frac{S[(R,U),(T,V)]}{S([R,T],[U,V])} \quad .$$

This rule has no analogue in the sequent calculus. But it is clearly sound because we can derive it:

Proposition 3.14. The medial rule is derivable for $\{\mathsf{c}{\downarrow}, \mathsf{w}{\downarrow}\}$. Dually, the medial rule is derivable for $\{\mathsf{c}{\uparrow}, \mathsf{w}{\uparrow}\}$.

Proof. The medial rule is derivable as follows (or dually):

$$
\mathsf{c}\!\downarrow \frac{\mathsf{w}\!\downarrow \frac{\mathsf{w}\!\downarrow \frac{\mathsf{w}\!\downarrow \frac{\mathsf{w}\!\downarrow \frac{S[(R,U),(T,V)]}{S[(R,U),(T,[U,V])]}}{S[(R,U),([R,T],[U,V])]}}{S[(R,[U,V]),([R,T],[U,V])]}}{S[([R,T],[U,V]),([R,T],[U,V])]}}{S([R,T],[U,V])} \quad .
$$

□

An analogue to the medial rule has also been considered by Došen and Petrić as a composite arrow in the free bicartesian category, cf. the end of Section 4 in [10]. It is composed of four projections and a pairing of identities (or dually) in the same way as medial is derived using four weakenings and a contraction in the proof above.

We define atomic variants of the rules for identity, cut, weakening and contraction from system SKSg shown in Figure 4.

$$
\mathsf{ai}\!\downarrow \frac{S\{\mathsf{t}\}}{S[a,\bar{a}]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\mathsf{f}\}}
$$

$$
\mathsf{aw}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{a\}} \qquad \mathsf{aw}\!\uparrow \frac{S\{a\}}{S\{\mathsf{t}\}}
$$

$$
\mathsf{ac}\!\downarrow \frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}\!\uparrow \frac{S\{a\}}{S(a,a)}
$$

Figure 4: Atomic identity, cut, weakening and contraction

Theorem 3.15. The rules $\mathsf{i}\!\downarrow$, $\mathsf{w}\!\downarrow$ and $\mathsf{c}\!\downarrow$ are derivable for $\{\mathsf{ai}\!\downarrow, \mathsf{s}\}$, $\{\mathsf{aw}\!\downarrow\}$ and $\{\mathsf{ac}\!\downarrow, \mathsf{m}\}$, respectively. Dually, the rules $\mathsf{i}\!\uparrow$, $\mathsf{w}\!\uparrow$ and $\mathsf{c}\!\uparrow$ are derivable for $\{\mathsf{ai}\!\uparrow, \mathsf{s}\}$, $\{\mathsf{aw}\!\uparrow\}$ and $\{\mathsf{ac}\!\uparrow, \mathsf{m}\}$, respectively.

Proof. I will show derivability of the rules $\{\mathsf{i}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{c}\!\downarrow\}$ for the respective systems. The proof of derivability of their co-rules is dual.

Given an instance of one of the following rules:

$$
\mathsf{i}\!\downarrow \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \quad , \qquad \mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}} \quad , \qquad \mathsf{c}\!\downarrow \frac{S[R,R]}{S\{R\}} \quad ,
$$

construct a new derivation by structural induction on R:

1. R is an atom. Then the instance of the general rule is also an instance of its atomic form.

2. $R = \mathsf{t}$ or $R = \mathsf{f}$. Then the instance of the general rule is an instance of the equivalence rule, with the only exception of weakening in case that $R = \mathsf{t}$. Then this instance of weakening can be replaced by

$$
= \frac{S\{\mathsf{f}\}}{\mathsf{s}\,\dfrac{S([\mathsf{t},\mathsf{t}],\mathsf{f})}{= \dfrac{S[\mathsf{t},(\mathsf{t},\mathsf{f})]}{S\{\mathsf{t}\}}}} \quad .
$$

3. $R = [P, Q]$. Apply the induction hypothesis respectively on

$$
\mathsf{i}{\downarrow}\frac{\mathsf{i}{\downarrow}\dfrac{S\{\mathsf{t}\}}{S[Q,\bar{Q}]}}{\mathsf{s}\,\dfrac{S([P,\bar{P}],[Q,\bar{Q}])}{\mathsf{s}\,\dfrac{S[Q,([P,\bar{P}],\bar{Q})]}{S[P,Q,(\bar{P},\bar{Q})]}}} \quad , \qquad
= \frac{S\{\mathsf{f}\}}{\mathsf{w}{\downarrow}\dfrac{S[\mathsf{f},\mathsf{f}]}{\mathsf{w}{\downarrow}\dfrac{S[\mathsf{f},Q]}{S[P,Q]}}} \quad , \qquad
\mathsf{c}{\downarrow}\frac{S[P,P,Q,Q]}{\mathsf{c}{\downarrow}\dfrac{S[P,P,Q]}{S[P,Q]}} \quad .
$$

4. $R = (P, Q)$. Apply the induction hypothesis respectively on

$$
\mathsf{i}{\downarrow}\frac{\mathsf{i}{\downarrow}\dfrac{S\{\mathsf{t}\}}{S[Q,\bar{Q}]}}{\mathsf{s}\,\dfrac{S([P,\bar{P}],[Q,\bar{Q}])}{\mathsf{s}\,\dfrac{S[([P,\bar{P}],Q),\bar{Q}]}{S[(P,Q),\bar{P},\bar{Q}]}}} \quad , \qquad
= \frac{S\{\mathsf{f}\}}{\mathsf{w}{\downarrow}\dfrac{S(\mathsf{f},\mathsf{f})}{\mathsf{w}{\downarrow}\dfrac{S(\mathsf{f},Q)}{S(P,Q)}}} \quad , \qquad
\mathsf{m}\,\frac{S[(P,Q),(P,Q)]}{\mathsf{c}{\downarrow}\dfrac{S([P,P],[Q,Q])}{\mathsf{c}{\downarrow}\dfrac{S([P,P],Q)}{S(P,Q)}}} \quad .
$$

□

We now define the local system SKS to be obtained from SKSg by restricting identity, cut, weakening and contraction to atomic form and adding medial, i.e.

$$
\mathsf{SKS} = \{\mathsf{ai}{\downarrow}, \mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{m}, \mathsf{aw}{\downarrow}, \mathsf{aw}{\uparrow}, \mathsf{ac}{\downarrow}, \mathsf{ac}{\uparrow}\} \quad .
$$

Theorem 3.16. System SKS and system SKSg are strongly equivalent.

Proof. Derivations in SKSg are translated to derivations in SKS by Theorem 3.15, and vice versa by Proposition 3.14. □

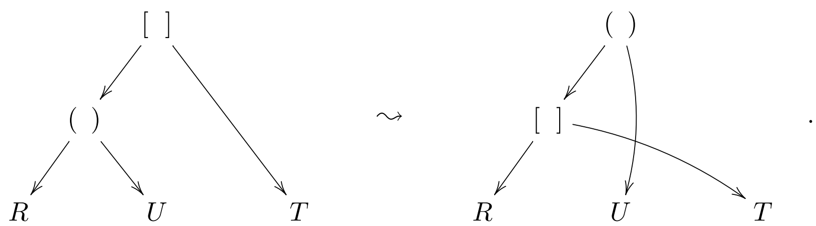
Thus, all results obtained for the global system, in particular the correspondence with the sequent calculus and admissibility of the up-rules, also hold for the local system. By removing the up-rules from system SKS we obtain system KS, i.e.

$$
\mathsf{KS} = \{\mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{m}, \mathsf{aw}{\downarrow}, \mathsf{ac}{\downarrow}\} \quad .
$$

Theorem 3.17. System KS and system KSg are strongly equivalent.

Proof. As the proof of Theorem 3.16. □

In system SKS, no rule requires duplicating formulas of arbitrary size. The atomic rules only need to duplicate, erase or compare atoms. The rules switch and medial do involve formulas of arbitrary size, just like the equations for associativity, commutativity and units. In the switch rule, for example, the formulas R, T and U are of arbitrary size. But applying those rules or equations does not require inspecting those formulas, so they are *local*. To see this, consider formulas represented as binary trees in the obvious way. Then the switch rule just changes the marking of two nodes and exchanges two pointers:



The same is true for medial. The equivalence rule clearly is not local, however, it is easily replaced by separate rules for associativity, commutativity and units which are local.

The informal notion of locality depends on the representation of formulas. Rules that are local for one representation may not be local when another representation is used. For example, the switch rule is local when formulas are represented as trees, but it is not local when formulas are represented as strings.

For the propositional case we can now give a candidate for a representation-independent definition of locality. In the terminology of term rewriting, local rules are very special: they are *non-erasing*, meaning that the variables occuring in the left-hand-side are exactly those that occur in the right-hand-side, and *left-linear* as well as *right-linear*, meaning that in both left- and right-hand-side there are no multiple occurrences of variables.

4 Predicate Logic

In this section I extend the deductive systems and the results about them from the previous section to predicate logic. The use of deep inference allows to design these systems in such a way that each rule corresponds to an implication from premise to conclusion, which is not true in the sequent calculus. Also, checking the eigenvariable conditions in this system does not require checking the entire context, in contrast to the R∀ rule in the sequent calculus. This allows to formulate a deduction theorem which does not have an analogue in the one-sided sequent calculus.

This section is structured as the previous one: after some basic definitions I present system SKSgq, a set of inference rules for classical predicate logic. I then extend the translations from the previous section, which establishes soundness and completeness with respect to classical predicate logic as well as cut admissibility. In the following I obtain an equivalent system, named SKSq, in which

$$\mathsf{u}\!\downarrow \frac{S\{\forall x[R,T]\}}{S[\forall xR,\exists xT]} \qquad \mathsf{u}\!\uparrow \frac{S(\exists xR,\forall xT)}{S\{\exists x(R,T)\}}$$

$$\mathsf{n}\!\downarrow \frac{S\{R[x/t]\}}{S\{\exists xR\}} \qquad \mathsf{n}\!\uparrow \frac{S\{\forall xR\}}{S\{R[x/t]\}}$$

Figure 5: Quantifier rules of System SKSgq

identity, cut, weakening and contraction are reduced to atomic form. The resulting system is local except for the rules that instantiate variables or check for free occurrences of a variable.

4.1 A Deep Inference System

We start with some basic definitions.

Definition 4.1. *Variables* are denoted by x and y. Terms are defined as usual in first-order predicate logic. The *formulas* of the language KSq are just like the formulas for propositional logic except that 1) instead of propositional variables they contain expressions of the form $p(t_1,\ldots,t_n)$, where p is a *predicate symbol* of *arity* n and $t_1,\ldots,t_n$ are terms, and 2) they may contain existential and universal quantifiers $\exists x$ and $\forall x$. The definition of the *negation* $\bar{S}$ of a formula S is extended as usual by $\overline{\exists xR} = \forall x\bar{R}$ and $\overline{\forall xR} = \exists x\bar{R}$. The notions of *formula context* and *subformula* are defined in the same way as in the propositional case.

Definition 4.2. Formulas are *syntactically equivalent* modulo the smallest congruence induced by the laws given in Definition 2.4 and the following laws:

Variable Renaming	$\forall xR = \forall yR[x/y]$ $\exists xR = \exists yR[x/y]$	if y is not free in R
Vacuous Quantifier	$\forall yR = \exists yR = R$	if y is not free in R

We obtain system SKSgq, a symmetric system for predicate logic, by adding the quantifier rules shown in Figure 5 to system SKSg, i.e.

$$\mathsf{SKSgq} = \mathsf{SKSg} \cup \{\mathsf{u}\!\downarrow,\mathsf{u}\!\uparrow,\mathsf{n}\!\downarrow,\mathsf{n}\!\uparrow\} \quad .$$

The rules $\mathsf{u}\!\downarrow$ and $\mathsf{u}\!\uparrow$ follow a scheme or recipe due to Guglielmi [12], which also yields the switch rule and ensures atomicity of cut and identity not only for classical logic but also for several other logics. The $\mathsf{u}\!\downarrow$ rule corresponds to the

$$\mathsf{R}\exists \frac{\vdash \Phi, A[x/t]}{\vdash \Phi, \exists x A} \qquad \mathsf{R}\forall \frac{\vdash \Phi, A[x/y]}{\vdash \Phi, \forall x A}$$

Proviso: y is not free in the conclusion of $\mathsf{R}\forall$.

Figure 6: Quantifier rules of $\mathsf{GS1}$

$\mathsf{R}\forall$ rule in $\mathsf{GS1}$, shown in Figure 6. We could equivalently replace it by

$$\mathsf{uv}\downarrow \frac{S\{\forall x[R,T]\}}{S[\forall x R, T]} \text{ if } x \text{ is not free in } T.$$

In the sequent calculus, going up, the $\mathsf{R}\forall$ rule removes a universal quantifier from a formula to allow other rules to access this formula. In system SKSgq, inference rules apply deep inside formulas, so there is no need to remove the quantifier: it can be moved out of the way using the rule $\mathsf{u}\downarrow$ and it vanishes once the proof is complete because of the equation $\forall x \mathsf{t} = \mathsf{t}$.

The rule $\mathsf{n}\downarrow$ corresponds to $\mathsf{R}\exists$. As usual, the substitution operation requires t to be free for x in R: quantifiers in R do not capture variables in t. The term t is not required to be free for x in $S\{R\}$: quantifiers in S may capture variables in t.

4.2 Correspondence to the Sequent Calculus

We extend the translations between SKSg and $\mathsf{GS1p}$ to translations between SKSgq and $\mathsf{GS1}$. System $\mathsf{GS1}$ is system $\mathsf{GS1p}$ extended by the rules shown in Figure 6.

From the Sequent Calculus to the Calculus of Structures

Theorem 4.3. For every derivation $\begin{matrix}\Sigma_1 \cdots \Sigma_h \\ \bigtriangledown \\ \Sigma\end{matrix}$ in $\mathsf{GS1}{+}\mathsf{Cut}$ there exists a formula $P\{\Sigma_1,\dots,\Sigma_h\}$ built from $\Sigma_1,\dots,\Sigma_h$ using only conjunction and universal quantification, and a derivation $\begin{matrix} P\{\Sigma_1,\dots,\Sigma_h\} \\ \Big\| \, \mathsf{SKSgq} \setminus \{\mathsf{w}\uparrow,\mathsf{c}\uparrow,\mathsf{u}\uparrow,\mathsf{n}\uparrow\} \\ \Sigma \end{matrix}$ with the same number of cuts.

Proof. The proof is mostly similar to the proof of Theorem 3.6. The difference are two more inductive cases, one for $\mathsf{R}\exists$, which is easily translated into an $\mathsf{n}\downarrow$, and one for $\mathsf{R}\forall$, which is shown here:

$$
\begin{array}{c}
\Sigma_1 \;\cdots\; \Sigma_h \\
\bigtriangledown \\
\mathsf{R}\forall \, \dfrac{\vdash \Phi, A[x/y]}{\vdash \Phi, \forall x A}
\end{array} \quad .
$$

By induction hypothesis we obtain a derivation Δ from which we build

$$
\begin{array}{rc}
 & \forall y P\{\Sigma_1, \ldots, \Sigma_h\} \\
 & \forall y\{\Delta\} \Big\Vert \, \mathsf{SKSgq} \setminus \{\mathsf{w}\uparrow, \mathsf{c}\uparrow, \mathsf{u}\uparrow, \mathsf{n}\uparrow\} \\
 & \forall y[\Phi, A[x/y]] \\
\mathsf{u}\downarrow & \overline{[\exists y \Phi, \forall y A[x/y]]} \\
= & \overline{[\Phi, \forall y A[x/y]]} \\
= & \overline{[\Phi, \forall x A]}
\end{array} \quad ,
$$

where in the lower instance of the equivalence rule y is not free in $\forall x A$ and in the upper instance of the equivalence rule y is not free in Φ: both due to the proviso of the $\mathsf{R}\forall$ rule. ☐

Corollary 4.4.

1. If a sequent Σ has a proof in $\mathsf{GS1} + \mathsf{Cut}$ then Σ has a proof in the system $\mathsf{SKSgq} \setminus \{\mathsf{w}\uparrow, \mathsf{c}\uparrow, \mathsf{u}\uparrow, \mathsf{n}\uparrow\}$.
2. If a sequent Σ has a proof in $\mathsf{GS1}$ then Σ has a proof in the system $\mathsf{SKSgq} \setminus \{\mathsf{i}\uparrow, \mathsf{w}\uparrow, \mathsf{c}\uparrow, \mathsf{u}\uparrow, \mathsf{n}\uparrow\}$.

From the Calculus of Structures to the Sequent Calculus

Lemma 4.5. For every two formulas A, B and every formula context $C\{\ \}$ there exists a derivation
$$
\begin{array}{c}
\vdash A, \bar{B} \\
\bigtriangledown \\
\vdash C\{A\}, \overline{C\{B\}}
\end{array}
$$
in $\mathsf{GS1}$.

Theorem 4.6. For every derivation $\begin{array}{c} Q \\ \Big\Vert \, \mathsf{SKSgq} \\ P \end{array}$ there exists a derivation $\begin{array}{c} \vdash Q \\ \bigtriangledown \\ \vdash P \end{array}$ in $\mathsf{GS1} + \mathsf{Cut}$.

Proof. The proof is an extension of the proof of Theorem 3.9. The base cases are the same, in the inductive cases the existence of Δ_1 follows from Lemma 4.5. Corresponding to the rules for quantifiers, there are four additional inductive cases, which are simple. We show the case for

$$\mathsf{u}\!\downarrow \frac{S\{\forall x[R,T]\}}{S[\forall xR, \exists xT]} \quad \text{for which we have} \quad \mathsf{R}\vee \frac{\mathsf{R}\forall \frac{\mathsf{R}\exists \frac{\mathsf{R}\exists \frac{\mathsf{R}\wedge \frac{\mathsf{Ax}\frac{}{\vdash R, \bar{R}} \quad \mathsf{Ax}\frac{}{\vdash T, \bar{T}}}{\vdash R, T, \bar{R} \wedge \bar{T}}}{\vdash R, \exists xT, \bar{R} \wedge \bar{T}}}{\vdash R, \exists xT, \exists x(\bar{R} \wedge \bar{T})}}{\vdash \forall xR, \exists xT, \exists x(\bar{R} \wedge \bar{T})}}{\vdash \forall xR \vee \exists xT, \exists x(\bar{R} \wedge \bar{T})} \quad .$$

□

Corollary 4.7. If a formula S has a proof in SKSgq then $\vdash S$ has a proof in GS1 + Cut.

4.3 Soundness, Completeness and Cut Admissibility

Just like in the propositional case, soundness and completeness of SKSgq, i.e. the fact that a formula has a proof in SKSgq if and only if it is valid, follows from soundness and completeness of GS1 by Corollaries 4.4 and 4.7.

All inference rule in GS1 are sound in the sense that the validity of the premise implies the validity of the conclusion. For system SKSgq something more is true: for each inference rule the premise implies the conclusion. This is not true for system GS1: the premise of the R$\forall$ rule does not imply its conclusion. The R$\forall$ rule is the only rule in GS1 with this behaviour.

The "strong soundness" of inference rules in system SKSgq relies on the fact that, by dropping the restrictions of the sequent calculus, we can pull out a universal quantifier going up in the u$\downarrow$ rule instead of having to drop it, as happens in the R$\forall$ rule. As a consequence we can prove a deduction theorem which does not have an analogue in the one-sided sequent calculus:

Theorem 4.8 (Deduction Theorem)**.**

There is a derivation $\begin{array}{c} T \\ \Vert \text{SKSgq} \\ R \end{array}$ if and only if there is a proof $\begin{array}{c} \overline{\Vert} \text{SKSgq} \\ [\bar{T}, R] \end{array}$.

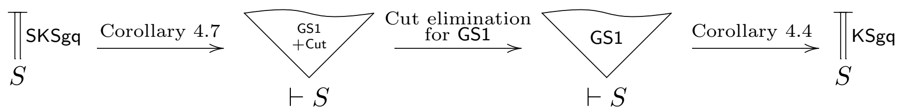
The proof is the same as the proof of Theorem 3.11 on page 11. Note that this proof does not work for the sequent calculus because adding to the context of a derivation can violate the proviso of the R$\forall$ rule.

Just like in the propositional case, the up-rules of the symmetric system are admissible. By removing them from SKSgq we obtain the asymmetric, cut-free system KSgq, i.e.

$$\mathsf{KSgq} = \mathsf{KSg} \cup \{\mathsf{u}\!\downarrow, \mathsf{n}\!\downarrow\} \quad .$$

Theorem 4.9 (Cut Elimination)**.** The rules i$\uparrow$, w$\uparrow$, c$\uparrow$, u$\uparrow$ and n$\uparrow$ are admissible for system KSgq. Put differently, the systems SKSgq and KSgq are equivalent.

Proof.



□

4.4 Atomicity and Locality

Just like in the propositional case, we can reduce identity, cut, weakening and contraction to their atomic forms. In order to reduce contraction to atomic form, we need to add the following local rules:

$$\mathsf{m_1}\!\downarrow \frac{S[\exists x R, \exists x T]}{S\{\exists x [R,T]\}} \qquad \mathsf{m_1}\!\uparrow \frac{S\{\forall x (R,T)\}}{S(\forall x R, \forall x T)}$$

$$\mathsf{m_2}\!\downarrow \frac{S[\forall x R, \forall x T]}{S\{\forall x [R,T]\}} \qquad \mathsf{m_2}\!\uparrow \frac{S\{\exists x (R,T)\}}{S(\exists x R, \exists x T)} \qquad .$$

Like medial, they have no analogues in the sequent calculus. In system SKSgq, and similarly in the sequent calculus, the corresponding inferences are made using contraction and weakening:

Proposition 4.10. The rules $\{\mathsf{m_1}\!\downarrow, \mathsf{m_2}\!\downarrow\}$ are derivable for $\{\mathsf{c}\!\downarrow, \mathsf{w}\!\downarrow\}$. Dually, the rules $\{\mathsf{m_1}\!\uparrow, \mathsf{m_2}\!\uparrow\}$ are derivable for $\{\mathsf{c}\!\uparrow, \mathsf{w}\!\uparrow\}$.

Proof. We show the case for $\mathsf{m_1}\!\downarrow$, the other cases are similar or dual:

$$\mathsf{c}\!\downarrow \frac{\mathsf{w}\!\downarrow \dfrac{\mathsf{w}\!\downarrow \dfrac{S[\exists x R, \exists x T]}{S[\exists x R, \exists x [R,T]]}}{S[\exists x [R,T], \exists x [R,T]]}}{S\{\exists x [R,T]\}} \qquad .$$

□

Theorem 4.11. The rules $\mathsf{i}\!\downarrow$, $\mathsf{w}\!\downarrow$ and $\mathsf{c}\!\downarrow$ are derivable for $\{\mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{u}\!\downarrow\}$, $\{\mathsf{aw}\!\downarrow\}$ and $\{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m_1}\!\downarrow, \mathsf{m_2}\!\downarrow\}$, respectively. Dually, the rules $\mathsf{i}\!\uparrow$, $\mathsf{w}\!\uparrow$ and $\mathsf{c}\!\uparrow$ are derivable for $\{\mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{u}\!\uparrow\}$, $\{\mathsf{aw}\!\uparrow\}$ and $\{\mathsf{ac}\!\uparrow, \mathsf{m}, \mathsf{m_1}\!\uparrow, \mathsf{m_2}\!\uparrow\}$, respectively.

Proof. The proof is an extension of the proof of Theorem 3.15 by the inductive cases for the quantifiers. Given an instance of one of the following rules:

$$\mathsf{i}\!\downarrow \frac{S\{\mathsf{t}\}}{S[R,\bar{R}]} \quad , \qquad \mathsf{w}\!\downarrow \frac{S\{\mathsf{f}\}}{S\{R\}} \quad , \qquad \mathsf{c}\!\downarrow \frac{S[R,R]}{S\{R\}} \quad ,$$

construct a new derivation by structural induction on R:

1. $R = \exists x T$. Apply the induction hypothesis respectively on

$$
\mathsf{u}\!\downarrow \frac{\mathsf{i}\!\downarrow \frac{= \frac{S\{\mathsf{t}\}}{S\{\forall x \mathsf{t}\}}}{S\{\forall x[T,\bar{T}]\}}}{S[\exists xT, \forall x\bar{T}]} \quad , \qquad \mathsf{w}\!\downarrow \frac{= \frac{S\{\mathsf{f}\}}{S\{\exists x \mathsf{f}\}}}{S\{\exists xT\}} \quad , \qquad \mathsf{c}\!\downarrow \frac{\mathsf{m_1}\!\downarrow \frac{S[\exists xT, \exists xT]}{S\{\exists x[T,T]\}}}{S\{\exists xT\}} \quad .
$$

2. $R = \forall xT$. Apply the induction hypothesis respectively on

$$
\mathsf{u}\!\downarrow \frac{\mathsf{i}\!\downarrow \frac{= \frac{S\{\mathsf{t}\}}{S\{\forall x \mathsf{t}\}}}{S\{\forall x[T,\bar{T}]\}}}{S[\forall xT, \exists x\bar{T}]} \quad , \qquad \mathsf{w}\!\downarrow \frac{= \frac{S\{\mathsf{f}\}}{S\{\forall x \mathsf{f}\}}}{S\{\forall xT\}} \quad , \qquad \mathsf{c}\!\downarrow \frac{\mathsf{m_2}\!\downarrow \frac{S[\forall xT, \forall xT]}{S\{\forall x[T,T]\}}}{S\{\forall xT\}} \quad .
$$

□

We now obtain system SKSq from SKSgq by restricting identity, cut, weakening and contraction to atomic form and adding the medial rules, i.e.

$$
\mathsf{SKSq} = \mathsf{SKS} \cup \{\mathsf{u}\!\downarrow, \mathsf{u}\!\uparrow, \mathsf{n}\!\downarrow, \mathsf{n}\!\uparrow\} \cup \{\mathsf{m_1}\!\downarrow, \mathsf{m_2}\!\downarrow, \mathsf{m_1}\!\uparrow, \mathsf{m_2}\!\uparrow\} \quad .
$$

As in all the systems considered, the up-rules are admissible and hence system

$$
\mathsf{KSq} = \mathsf{KS} \cup \{\mathsf{u}\!\downarrow, \mathsf{n}\!\downarrow, \} \cup \{\mathsf{m_1}\!\downarrow, \mathsf{m_2}\!\downarrow\}
$$

is complete.

Theorem 4.12.

1. System SKSq and system SKSgq are strongly equivalent.
2. System KSq and system KSgq are strongly equivalent.

Proof. Derivations in SKSgq are translated to derivations in SKSq by Theorem 4.11, and vice versa by Proposition 4.10. The same holds for KSgq and KSq. □

Thus, soundness, completeness and cut admissibility as obtained for system SKSgq also hold for system SKSq.

As we have seen in the previous section, the technique of reducing contraction to atomic form to obtain locality also works in the case of predicate logic: the general contraction rule is equivalently replaced by local rules, namely $\{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{m_1}\!\downarrow, \mathsf{m_2}\!\downarrow\}$.

However, there are other sources of globality in system SKSq. One is the condition on the quantifier equations:

$$
\forall yR = \exists yR = R \qquad \text{where } y \text{ is not free in } R.
$$

To add or remove a quantifier, a formula of arbitrary size has to be checked for occurrences of the variable y.

Another is the $\mathsf{n}\!\downarrow$ rule, in which a term t of arbitrary size has to be copied into an arbitrary number of occurrences of x in R. It is global for two distinct reasons: 1) the arbitrary size of t and 2) the arbitrary number of occurrences of x in R. The arbitrary size of term t can be dealt with, since $\mathsf{n}\!\downarrow$ can easily be derived and thus replaced by the following two rules:

$$\mathsf{n}_1\!\downarrow \frac{S\{\exists y_1 \ldots \exists y_n R[x/f(y_1,...,y_n)]\}}{S\{\exists x R\}} \qquad \text{and} \qquad \mathsf{n}_2\!\downarrow \frac{S\{R\}}{S\{\exists x R\}} \quad ,$$

where f is a function symbol of arity n. Still, rule $\mathsf{n}_1\!\downarrow$ is not local because of the arbitrary number of occurrences of x in R.

Is it possible to obtain a local system for first-order predicate logic? It is certainly possible if we were to add new symbols to the language of predicate logic. We could introduce substitution operators together with rules that explicitly handle the instantiation of a variable in a formula piece by piece. However, extending the language for that purpose seems ad-hoc and would not achieve our goal of having locality *inside* the proof-theoretical system, i.e. inside a system which should be simple and should allow to comfortably study properties like cut admissibility.

5 Conclusions

We have seen deductive systems for classical propositional and predicate logic in the calculus of structures. They are sound and complete, and the cut rule is admissible. In contrast to sequent systems, their rules apply *deep* inside formulas, and derivations enjoy a top-down *symmetry* which allows to dualise them.

Those features allow to reduce the cut, weakening and contraction to atomic form, which is not possible in the sequent calculus. This leads to *local* rules, i.e. rules that do not require the inspection of expressions of arbitrary size. Apart from the treatment of variables in the system for predicate logic, the systems that I presented are entirely local.

Compared to the sequent calculus, there is much more freedom in applying inference rules in the calculus of structures. This freedom allows to easily embed not only the sequent calculus itself and natural deduction, but also methods known from automated theorem proving. Resolution [24], for instance, can straightforwardly be seen as a strategy for proof search in system SKS [15]. The calculus of structures could thus be used to study these methods in a unified formalism.

The freedom in applying inference rules is a mixed blessing. Compared to the sequent calculus it allows for shorter proofs, cf. [14], but the greater non-determinism in proof search also makes it harder to find proofs. It will be interesting to see how to restrict this non-determinism by finding a suitable notion of goal-driven proof like the notion of *uniform proofs* by Miller et al. [22]. Since the sequent calculus can be seen as a strategy in the calculus of structures and uniformity can be seen as a strategy in the sequent calculus it seems promising to try to obtain a more general notion of goal-drivenness.

Some progress has already been made in restricting the non-determinism in system SKS by so-called *decomposition theorems* [5], which provide notions of normal form for derivations in a natural way, namely by restricting the choice of which inference rules to apply. Finding more decomposition theorems is an interesting task for future research since it turns out that many proof theoretical phenomena can be stated as a suitable decomposition theorem, like Herbrand's Theorem, Craig interpolation or cut admissibility. There is also work by Kahramanoğullari [18] on reducing non-determinism and implementations.

The decomposition of the contraction rule into atomic contraction and medial has been fruitful for the work by Lamarche and Straßburger [20, 30] who develop notions of classical proof net and categorical axiomatisations for classical proofs. McKinley [21] also gives a categorical axiomatisation for proofs in classical predicate logic which is partly inspired by the shape of the medial rules.

An interesting question is whether there are local systems for non-classical logics. In the case of modal logic the reducibility of rules to atomic form straightforwardly scales to the systems presented by Stewart and Stouppa in [27]. In the case of intuitionistic logic this is not so straightforward. Implication can not be expressed by disjunction and negation as in the classical case, we need it as a primitive connective in the system. The reduction of identity and cut still works [3], but we have yet to find a way to reduce contraction to atomic form in the presence of implication.

Acknowledgements
Most of this work has been accomplished while I was supported by the DFG Graduiertenkolleg 334. I would like to thank the members of the proof theory group in Dresden for providing an inspiring environment, especially Alessio Guglielmi. He also helped me with this paper in numerous ways. Steffen Hölldobler, Lutz Straßburger and Charles Stewart carefully read preliminary versions of this paper and made helpful suggestions.

Web Site
Information about the calculus of structures is available from the following URL:

http://alessio.guglielmi.name/res/cos/index.html .

References

[1] Franz Baader and Tobias Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1998.

[2] Andreas Blass. A game semantics for linear logic. *Annals of Pure and Applied Logic*, 56:183–220, 1992.

[3] Kai Brünnler. Minimal logic in the calculus of structures, 2003. Note. On the web at: http://www.iam.unibe.ch/~kai/Current/minimal.html.

[4] Kai Brünnler. Atomic cut elimination for classical logic. In M. Baaz and J. A. Makowsky, editors, *CSL 2003*, volume 2803 of *Lecture Notes in Computer Science*, pages 86–97. Springer-Verlag, 2003.

[5] Kai Brünnler. *Deep Inference and Symmetry in Classical Proofs*. PhD thesis, Technische Universität Dresden, September 2003.

[6] Kai Brünnler. Two restrictions on contraction. *Logic Journal of the IGPL*, 11(5):525–529, 2003.

[7] Kai Brünnler. Cut elimination inside a deep inference system for classical predicate logic. *Studia Logica*, 82(1):51–71, 2006.

[8] Kai Brünnler and Alwen Fernanto Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347–361. Springer-Verlag, 2001.

[9] Paola Bruscoli. A purely logical account of sequentiality in proof search. In Peter J. Stuckey, editor, *Logic Programming, 18th International Conference*, volume 2401 of *Lecture Notes in Artificial Intelligence*, pages 302–316. Springer-Verlag, 2002.

[10] Kosta Došen and Zoran Petrić. Bicartesian coherence. *Studia Logica*, 71(3):331–353, 2002.

[11] Gerhard Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland Publishing Co., Amsterdam, 1969.

[12] Alessio Guglielmi. Recipe. Manuscript. http://www.wv.inf.tu-dresden.de/~guglielm/Research/Notes/AG2.pdf, 2002.

[13] Alessio Guglielmi. A system of interaction and structure. Technical Report WV-02-10, Technische Universität Dresden, 2002. To appear in ACM Transactions on Computational Logic.

[14] Alessio Guglielmi. Polynomial size deep-inference proofs instead of exponential size shallow-inference proofs. Manuscript. http://www.ki.inf.tu-dresden.de/~guglielm/res/notes/AG12.pdf, 2003.

[15] Alessio Guglielmi. Resolution in the calculus of structures. Manuscript. http://www.ki.inf.tu-dresden.de/~guglielm/res/notes/AG10.pdf, 2003.

[16] Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In L. Fribourg, editor, *CSL 2001*, volume 2142 of *Lecture Notes in Computer Science*, pages 54–68. Springer-Verlag, 2001.

[17] Alessio Guglielmi and Lutz Straßburger. A non-commutative extension of MELL. In Matthias Baaz and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2002*, volume 2514 of *LNAI*, pages 231–246. Springer-Verlag, 2002.

[18] Ozan Kahramanoğullari. Reducing nondeterminism in the calculus of structures. Manuscript, 2005.

[19] François Lamarche and Lutz Straßburger. Constructing free boolean categories. In *20th Annual IEEE Symposium on Logic in Computer Science (LICS' 05)*, pages 209–218, 2005.

[20] François Lamarche and Lutz Straßburger. Naming proofs in classical propositional logic. In Paweł Urzyczyn, editor, *Typed Lambda Calculi and Applications, TLCA 2005*, volume 3461 of *Lecture Notes in Computer Science*, pages 246–261. Springer-Verlag, 2005.

[21] Richard McKinley. Categorical models of first-order classical proofs. submitted for the degree of Doctor of Philosophy of the University of Bath, 2005.

[22] Dale Miller, Gopalan Nadathur, Frank Pfenning, and Andre Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51:125–157, 1991.

[23] W.V. Quine. *Mathematical Logic. Revised Edition.* Harvard University Press, Cambridge, 1981.

[24] John Alan Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12:23–41, 1965.

[25] Kurt Schütte. Schlussweisen-Kalküle der Prädikatenlogik. *Mathematische Annalen*, 122:47–65, 1950.

[26] Kurt Schütte. *Beweistheorie*. Springer-Verlag, 1960.

[27] Charles Stewart and Phiniki Stouppa. A systematic proof theory for several modal logics. In Renate Schmidt, Ian Pratt-Hartmann, Mark Reynolds, and Heinrich Wansing, editors, *Advances in Modal Logic*, volume 5 of *King's College Publications*, pages 309–333, 2005.

[28] Lutz Straßburger. MELL in the Calculus of Structures. *Theoretical Computer Science*, 309(1–3):213–285, 2003.

[29] Lutz Straßburger. A local system for linear logic. In Matthias Baaz and Andrei Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2002*, volume 2514 of *LNAI*, pages 388–402. Springer-Verlag, 2002.

[30] Lutz Straßburger. On the axiomatisation of boolean categories with and without medial. Manuscript, 2005.

[31] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, 1996.