Two Restrictions on Contraction

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Abstract

I show two simple limitations of sequent systems with multiplicative context treatment: contraction can be restricted neither to atoms nor to the bottom of a proof tree.

Keywords: sequent calculus, contraction, deep inference

1 Motivation

The motivation for the present work is to find out whether there is a sequent system that possesses certain properties of system SKS, a set of rules for classical propositional logic introduced in [1]. System SKS is not a sequent system, but is presented in a more general formalism, the *calculus of structures* [2]. In this formalism, an inference rule has only one premise: derivations are sequences of rule instances, not trees as in the sequent calculus. While the sequent calculus restricts the application of rules to the main connective of a formula, the calculus of structures is more expressive by admitting *deep inference*, meaning that rules can be applied anywhere inside formulae.

Similarly to sequent systems, system SKS has a contraction rule which, when seen bottom-up, duplicates a formula. This contraction rule can be restricted 1) to atoms and 2) to the bottom of a proof. Apart from contraction, no other rule duplicates formulae. The two restrictions on contraction thus respectively entail the following two interesting properties [1]:

1. Applying a rule may involve duplicating atoms, but not duplicating arbitrarily large non-atomic formulae.
2. Proofs can be separated into two phases (seen bottom-up): The lower phase only contains instances of contraction. The upper phase contains instances of the other rules, but no contraction. No formulae are duplicated in the upper phase.

The question is whether the extra expressive power of the calculus of structures is needed for these properties, or whether they can be obtained in sequent systems as well. In system G3cp [3], for example, contraction is admissible and can thus trivially be restricted to atoms or to the bottom of a proof. However, G3cp has an additive (or context-sharing) R$\wedge$-rule, so these restrictions on contraction do not entail the above mentioned interesting properties. Contraction is admissible, but additive rules such as R$\wedge$ implicitly duplicate formulae which may be non-atomic. Of course, R$\wedge$ is not eliminable. To obtain a proof separation similar to the one for system SKS, one would have to restrict contraction *and* R∧ to the bottom of a proof tree, which is not possible. Other sequent systems with an additive R∧-rule share these problems.

To answer the question whether there is a sequent system with the properties of system SKS, I thus consider systems with a multiplicative (or context-splitting) R∧-rule, exemplified by system GS1p with multiplicative context treatment [3], shown in Fig. 1.

$$\mathsf{Ax}\,\frac{}{\vdash A, \bar{A}} \qquad \mathsf{RC}\,\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \mathsf{RW}\,\frac{\vdash \Gamma}{\vdash \Gamma, A}$$

$$\mathsf{R}\vee\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B} \qquad \mathsf{R}\wedge\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B}$$

FIG. 1. GS1p with multiplicative context treatment

The rules Ax, RC and RW are respectively called *axiom*, *contraction* and *weakening*. Propositional variables p and their negations $\bar{p}$ are *atoms*, with the negation of the atom $\bar{p}$ defined to be p. Atoms are denoted by $a, b, \ldots$. Formulae, denoted by $A, B, \ldots$, are in *negation normal form*, meaning that they contain negation only on atoms. $\bar{A}$ denotes the negation normal form of the negation of formula A. A *derivation* (also called partial proof) is a tree of rule instances. A *proof* is a derivation where all leaves are axioms. In a derivation, *all contractions are at the bottom* if no contraction is applied above a rule different from contraction. An application of the contraction rule is said to be *atomic* if its principal formula is an atom. The *endsequent* of a derivation is the sequent at the root. The system GS1p *with atomic axiom* is GS1p with the formulas A and $\bar{A}$ in the axiom required to be atoms.

2 Restricting Contraction in the Sequent Calculus is Impossible

In the following, I will show that GS1p does not possess the properties of SKS.

PROPOSITION 2.1
There is a valid sequent that has no proof in multiplicative GS1p in which all contractions are atomic.

PROOF. Consider the following sequent:

$$\vdash a \wedge b, (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}) \quad . \tag{2.1}$$

There are no single atoms, so contraction cannot be applied. Each applicable rule leads to a premise that is not valid. ∎

PROPOSITION 2.2
There is a valid sequent that has no proof in multiplicative GS1p in which all contractions are at the bottom.

Proof. Consider the following sequent:

$$\vdash a \wedge a, \bar{a} \wedge \bar{a} \quad . \tag{2.2}$$

It suffices to show that, for any number of occurrences of the formulae $a \wedge a$ and $\bar{a} \wedge \bar{a}$, the sequent

$$\vdash a \wedge a, \ldots, a \wedge a, \bar{a} \wedge \bar{a}, \ldots, \bar{a} \wedge \bar{a} \tag{2.3}$$

is not provable in GS1p without contraction. Since the connective $\vee$ does not occur in this sequent, the only rules that can appear in contraction-free derivations with this endsequent are Ax, R$\wedge$ and RW. The only formulae that can appear in such derivations are $a \wedge a, \bar{a} \wedge \bar{a}, a$ and $\bar{a}$. Consequently, the only formulae that can appear in an axiom are the atoms a and $\bar{a}$. A leaf can thus be closed with an axiom only if it contains exactly two single atoms (as opposed to two non-atomic formulae).

We prove by induction on the size of the derivation that each such derivation has a leaf which contains at most one single atom. The base case is trivial. For the inductive case, consider a derivation $\mathcal{D}$. Remove a rule instance ρ from the top of $\mathcal{D}$, to obtain a derivation $\mathcal{D}'$. Let l be the leaf with the conclusion of ρ. By inductive hypothesis, $\mathcal{D}'$ has a leaf with at most one single atom. Assume that this leaf is l, otherwise the inductive step is trivial. The rule instance ρ can not be an axiom, because there is at most one single atom in l. If ρ is a weakening then the premise of ρ contains at most one single atom. If ρ is an instance of R$\wedge$ then the only single atom that may occur in the conclusion goes to one premise. The other premise contains at most one (i.e. exactly one) single atom. ■

A referee found a simpler proof of Proposition 2.2 by using the following fact:

Fact 2.3
If a sequent has a contraction-free proof in GS1p then it has a contraction-free proof in GS1p with atomic axiom.

This proof is as follows: consider proofs in GS1p with atomic axiom. By a trivial induction on the structure of the proof it follows that every contraction-free proof has an endsequent with at least two single atoms or at least one occurence of the connective $\vee$. Thus, sequent (2.3) has no contraction-free proof in GS1p with atomic axiom. By contrapositive of the above fact, it has no contraction-free proof in GS1p.

The reason for presenting the more complex proof is that it is more general: it applies to systems for which the above fact does not hold, e.g. GS1p with multiplicative R$\wedge$ and additive R$\vee$. In fact, the proofs of Propositions 2.1 and 2.2 rely on the multiplicative context treatment in the R$\wedge$-rule, but work regardless of whether the system in question is for propositional or for first-order predicate logic, whether it is two- or one-sided, whether or not rules for implication are in the system, whether it is related to G1 (explicit weakening) or G2 (weakening built into the axiom) and whether a multiplicative or additive version of the R$\vee$-rule is used. In those sequent systems, contraction can thus neither be restricted to atoms nor to the bottom of a proof. Consequently, those systems do not have the interesting properties of system SKS.

$$
\mathsf{c}\,\dfrac{\mathsf{c}\,\dfrac{\mathsf{m}\,\dfrac{\mathsf{R}\wedge\,\dfrac{\mathsf{R}\vee\,\dfrac{\mathsf{R}\wedge\,\dfrac{\vdash a,\bar{a} \quad \vdash b,\bar{b}}{\vdash a\wedge b,\bar{a},\bar{b}}}{\vdash a\wedge b,\bar{a}\vee\bar{b}} \qquad \mathsf{R}\vee\,\dfrac{\mathsf{R}\wedge\,\dfrac{\vdash a,\bar{a} \quad \vdash b,\bar{b}}{\vdash a\wedge b,\bar{a},\bar{b}}}{\vdash a\wedge b,\bar{a}\vee\bar{b}}}{\vdash a\wedge b, a\wedge b, (\bar{a}\vee\bar{b})\wedge(\bar{a}\vee\bar{b})}}{\vdash (a\vee a)\wedge(b\vee b), (\bar{a}\vee\bar{b})\wedge(\bar{a}\vee\bar{b})}}{\vdash (a\vee a)\wedge b, (\bar{a}\vee\bar{b})\wedge(\bar{a}\vee\bar{b})}}{\vdash a\wedge b, (\bar{a}\vee\bar{b})\wedge(\bar{a}\vee\bar{b})}
$$

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$$

FIG. 2. Proofs using deep inference and medial

3 Restricting Contraction by Using Deep Inference

To complete this exposition, I want to give an idea on how the sequents (2.1) and (2.2) are proved in SKS. Contraction can be restricted to the bottom of a proof, because it applies anywhere deep inside a formula. A corresponding rule in sequent calculus notation might look like

$$
\mathsf{c}\,\dfrac{\vdash \Gamma, F\{A\vee A\}}{\vdash \Gamma, F\{A\}} \quad ,
$$

where $F\{\ \}$ is a formula context. Contraction can be restricted to atoms because of deep inference and a rule which is called *medial*. A corresponding rule in sequent calculus notation might look like

$$
\mathsf{m}\,\dfrac{\vdash \Gamma, A\wedge C, B\wedge D}{\vdash \Gamma, (A\vee B)\wedge(C\vee D)} \quad .
$$

I do not want to suggest that those rules should be added to sequent systems, I just present them as sequent calculus rules to avoid going into technical details of system SKS, which can be found in [1]. Using deep inference and medial, we can prove the sequents (2.1) and (2.2) as shown in Fig. 2. Note that in both proofs all contractions are atomic and at the bottom.

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