Syntactic Cut Elimination for an Infinitary System for Common Knowledge

Kai Brünnler
(joint work with Thomas Studer)

Outline

Why Sequent Calculus?

Good Sequent Calculus =

1. axiomatisation

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1. axiomatisation
2. decision procedure
3. tool for establishing properties of the logic
4. a setting for studying proofs

Crash Course in Ordinals

well-order: strict linear order without infinite descending chains
ordinal: (morally) an equivalence class of well-orders

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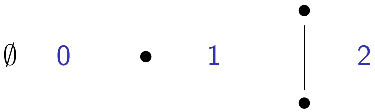

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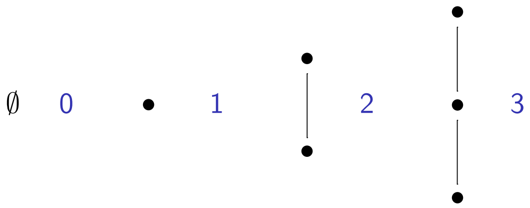

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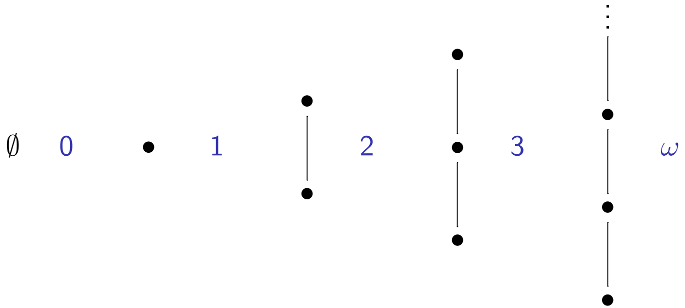

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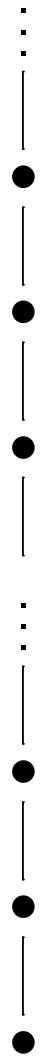

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The *natural sum* $\alpha \# \beta$ can be thought of as the sum "which adds in the right order" and thus is order-preserving and commutative. For example

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$\varphi_0 \beta := \omega^\beta$
$\varphi_\alpha \beta :=$ the β-th common fixpoint of the functions φ_γ for $\gamma < \alpha$.

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For example

$$\varphi_0 0 = 1 \qquad \varphi_0 1 = \omega \qquad \varphi_0 \alpha = \omega^\alpha$$

$$\varphi_1 0 = \mu\alpha.\omega^\alpha = \varepsilon_0 \qquad \varphi_1 \alpha = \varepsilon_\alpha \qquad \varphi_2 0 = \mu\alpha.\varphi_1 \alpha = \mu\alpha.\varepsilon_\alpha$$

The Logic of Common Knowledge

Formulas

$$A ::= p \mid \bar{p} \mid (A \vee A) \mid (A \wedge A) \mid \Diamond_i A \mid \Box_i A \mid \circledast A \mid \boxast A$$

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Everybody knows

$$\Box A = \Box_1 A \wedge \ldots \wedge \Box_h A \quad \text{and} \quad \Diamond A = \Diamond_1 A \vee \ldots \vee \Diamond_h A.$$

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Notation

$$\Box^n A = \underbrace{\Box \ldots \Box}_{n-\text{times}} A$$

A Hilbert System

System $\mathsf{H_C}$

$$
\begin{array}{ll}
\text{(K)} & \Box_i A \wedge \Box_i (A \supset B) \supset \Box_i B \\
\text{(CCL)} & \boxast A \supset (\Box A \wedge \Box \boxast A)
\end{array}
$$

$$
\text{(IND)}\ \frac{B \supset (\Box A \wedge \Box B)}{B \supset \boxast A} \qquad \text{(NEC)}\ \frac{A}{\Box_i A}
$$

$$
\text{(MP)}\ \frac{A \quad A \supset B}{B}
$$

A Shallow Sequent System

(Alberucci and Jäger 2005)

System $\mathsf{G_C}$

$$\Gamma, p, \bar{p} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\Box_i\,\frac{\Gamma, \circledast\Delta, A}{\Diamond_i\Gamma, \circledast\Delta, \Box_i A, \Sigma}$$

$$\boxast\,\frac{\Gamma, \Box^k A \quad \text{for all } k \geq 1}{\Gamma, \boxast A} \qquad \circledast\,\frac{\Gamma, \circledast A, \Diamond A}{\Gamma, \circledast A}$$

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The Problem for Cut Elimination

$$\text{cut}\;\dfrac{\Box_i\;\dfrac{\begin{matrix}\pi_1\\ A, \Gamma, \diamondast\bar{B}\end{matrix}}{\Box_i A, \Diamond_i \Gamma, \Sigma, \diamondast\bar{B}} \qquad \boxast\;\dfrac{\vdots \qquad \begin{matrix}\pi_{2k}\\ \Box^k B, \Delta\end{matrix} \qquad \vdots}{\boxast B, \Delta}\;\; 1 \leq k < \omega}{\Box_i A, \Diamond_i \Gamma, \Sigma, \Delta}$$

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\qquad
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}{
\Box_i A, \Diamond_i \Gamma, \Sigma, \Delta
}\;\text{cut}
$$

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$$
\text{cut}\;\dfrac{\Box_i\;\dfrac{\overset{\pi_1}{A, \Gamma, \diamondast \bar{B}}}{\Box_i A, \Diamond_i \Gamma, \Sigma, \diamondast \bar{B}} \qquad \boxast\;\dfrac{\vdots \quad \overset{\pi_{2k}}{\Box^k B, \Delta} \quad \vdots}{\boxast B, \Delta}\;1 \leq k < \omega}{\Box_i A, \Diamond_i \Gamma, \Sigma, \Delta}
$$

$$
\rightsquigarrow \qquad \Box_i\;\dfrac{\text{cut}\;\dfrac{\overset{\pi_1}{A, \Gamma, \diamondast \bar{B}} \qquad \boxast\;\dfrac{\vdots \quad \overset{\pi_{2k}}{\Box^k B, \Delta} \quad \vdots}{\boxast B, \Delta}\;1 \leq k < \omega}{A, \Gamma, \Delta}}{\Box_i A, \Diamond_i \Gamma, \Sigma, \Diamond_i \Delta}
$$

Some Background on Deep Inference

Deep Inference in General

- Schütte 1950
- Guglielmi 2001

Deep Sequents in Particular

- Schütte 1968 (not in an inference system)
- Kashima 1994, Tanaka 2003
- Brünnler 2006
- Poggiolesi (PhD student in Florence)

Deep Sequents

Definition

A *(deep) sequent* is a finite multiset of formulas and boxed sequents. A *boxed sequent* is an expression $[\Gamma]_i$ where Γ is a sequent and i is an agent.

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Notation

As usual! The comma in the expression Γ, Δ is multiset union. A singleton is the same as its element. Sequents are written without curly braces:

$$A, [[B]_1, C, [D]_2]_3$$

instead of

$$\{A, [\{[\{B\}]_1, C, [\{D\}]_2\}]_3\} \qquad .$$

Fact

A sequent is always of the form

$$A_1, \ldots, A_m, [\Delta_1]_{i_1}, \ldots, [\Delta_n]_{i_n} \quad .$$

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Definition

The *corresponding formula* of the above sequent is

$$A_1 \vee \cdots \vee A_m \vee \Box_{i_1} D_1 \vee \cdots \vee \Box_{i_n} D_n \quad ,$$

where $D_1 \ldots D_n$ are the corresponding formulas of the sequents $\Delta_1 \ldots \Delta_n$.

Sequent Contexts

Definition

A *context* is a sequent with exactly one occurrence of the symbol $\{ \}$, the *hole*, which does not occur inside formulas. Such contexts are denoted by $\Gamma\{ \}$. The sequent $\Gamma\{\Delta\}$ is obtained by replacing $\{ \}$ in $\Gamma\{ \}$ by Δ.

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Example

$$\text{If} \qquad \Gamma\{\ \} = A, [[B]_1, \{\ \}]_3 \qquad \text{and} \qquad \Delta = C, [D]_2$$

then

$$\Gamma\{\Delta\} = A, [[B]_1, C, [D]_2]_3 \quad .$$

System $\mathsf{D_C}$: A Deep Sequent System

$$\Gamma\{p, \bar{p}\} \qquad \wedge\ \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \qquad \vee\ \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}$$

$$\Box_i\ \frac{\Gamma\{[A]_i\}}{\Gamma\{\Box_i A\}} \qquad \Diamond_i\ \frac{\Gamma\{\Diamond_i A, [\Delta, A]_i\}}{\Gamma\{\Diamond_i A, [\Delta]_i\}}$$

$$\boxast\ \frac{\Gamma\{\Box^k A\} \quad \text{for all } k \geq 1}{\Gamma\{\boxast A\}} \qquad \circledast\ \frac{\Gamma\{\circledast A, \Diamond^k A\}}{\Gamma\{\circledast A\}}$$

Formula rank

$$rk(p) = rk(\bar{p}) = 0$$
$$rk(A \wedge B) = rk(A \vee B) = max(rk(A), rk(B)) + 1$$
$$rk(\Box_i A) = rk(\Diamond_i A) = rk(A) + 1$$
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&rk(\boxast A) = rk(\diamond\!\!\!\ast A) = \omega + rk(A)
\end{aligned}
$$

Lemma (Some properties of the rank)

(i) $rk(A) = rk(\bar{A})$,
(ii) $rk(A) < \omega^2$,
(iii) for all $k < \omega$ *we have* $rk(\Box^k A) < rk(\boxast A)$.

Structural rules and cut

$$\mathsf{nec}\,\frac{\Gamma}{[\Gamma]_i} \qquad \mathsf{wk}\,\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{ctr}\,\frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}}$$

$$\mathsf{cut}\,\frac{\Gamma\{A\} \qquad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

Lemma (Admissibility of the structural rules)

For system $\mathsf{D_C}$ *the following hold:*
(i) The necessitation, weakening and contraction rules are depth- and cut-rank-preserving admissible.
(ii) All rules are depth- and cut-rank-preserving invertible.

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Lemma (Admissibility of the general identity axiom)

For all contexts $\Gamma\{\ \}$ *and all formulas* A *we have*
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Theorem (Embedding the Hilbert system)

If $\mathsf{H_C} \vdash A$ *then* $\mathsf{D_C} \vdash^{<\omega^2}_{\omega^2} A$.

Lemma (Reduction Lemma)

If there is a proof

$$\mathsf{cut}_\gamma \; \frac{\overset{\displaystyle \pi_1}{\bigtriangledown}\atop \Gamma\{A\} \qquad\qquad \overset{\displaystyle \pi_2}{\bigtriangledown}\atop \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$

with π_1 *and* π_2 *in* $\mathsf{D}_\mathsf{C} + \mathsf{cut}_{<\gamma}$ *then* $\mathsf{D}_\mathsf{C} \,\mathop{\vdash}\limits^{|\pi_1| \,\#\, |\pi_2|}_{\gamma}\, \Gamma\{\emptyset\}$ *.*

$(\wedge - \vee)$:

$$\mathsf{cut}_{\sigma+1}\ \dfrac{\wedge\ \dfrac{\begin{array}{c}\pi_{11}\\ \Gamma\{B\}\end{array} \qquad \begin{array}{c}\pi_{12}\\ \Gamma\{C\}\end{array}}{\Gamma\{B \wedge C\}} \qquad \vee\ \dfrac{\begin{array}{c}\pi_{21}\\ \Gamma\{\bar{B}, \bar{C}\}\end{array}}{\Gamma\{\bar{B} \vee \bar{C}\}}}{\Gamma\{\emptyset\}}$$

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$$\rightsquigarrow \qquad \mathsf{cut}_{\sigma}\ \dfrac{\begin{array}{c}\pi_{11}\\ \Gamma\{B\}\end{array} \qquad \mathsf{cut}_{\sigma}\ \dfrac{\mathsf{wk}\ \dfrac{\begin{array}{c}\pi_{12}\\ \Gamma\{C\}\end{array}}{\Gamma\{\bar{B}, C\}} \qquad \begin{array}{c}\pi_{21}\\ \Gamma\{\bar{B}, \bar{C}\}\end{array}}{\Gamma\{\bar{B}\}}}{\Gamma\{\emptyset\}}$$

$(\Box_i - \Diamond_i)$:

$$
\dfrac{
\Box_i \, \dfrac{\dfrac{\pi_{11}}{\Gamma\{[\Delta]_i, [A]_i\}}}{\Gamma\{[\Delta]_i, \Box_i A\}}
\qquad
\Diamond_i \, \dfrac{\dfrac{\pi_{21}}{\Gamma\{[\Delta, \bar{A}]_i, \Diamond_i \bar{A}\}}}{\Gamma\{[\Delta]_i, \Diamond_i \bar{A}\}}
}{\Gamma\{[\Delta]_i\}}\ \mathsf{cut}_{\sigma+1}
$$

$(\Box_i - \Diamond_i)$:

$$
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\qquad
\Diamond_i\,\dfrac{\begin{matrix}\pi_{21}\\ \Gamma\{[\Delta,\bar{A}]_i,\Diamond_i \bar{A}\}\end{matrix}}{\Gamma\{[\Delta]_i,\Diamond_i \bar{A}\}}
}{\Gamma\{[\Delta]_i\}}
$$

$\rightsquigarrow$

$$
\mathsf{cut}_{\sigma}\;\dfrac{
\mathsf{ctr}\,\dfrac{\mathsf{wk}^2\,\dfrac{\begin{matrix}\pi_{11}\\ \Gamma\{[\Delta]_i,[A]_i\}\end{matrix}}{\Gamma\{[\Delta,A]_i,[\Delta,A]_i\}}}{\Gamma\{[\Delta,A]_i\}}
\qquad
\mathsf{cut}_{\sigma+1}\,\dfrac{
\mathsf{wk},\Box_i\,\dfrac{\begin{matrix}\pi_{11}\\ \Gamma\{[\Delta]_i,[A]_i\}\end{matrix}}{\Gamma\{[\Delta,\bar{A}]_i,\Box_i A\}}
\qquad
\begin{matrix}\pi_{21}\\ \Gamma\{[\Delta,\bar{A}]_i,\Diamond_i \bar{A}\}\end{matrix}
}{\Gamma\{[\Delta,\bar{A}]_i\}}
}{\Gamma\{[\Delta]_i\}}
$$

$(\boxast - \diamondast)$:

$$
\dfrac{
  \boxast\;\dfrac{\vdots \qquad \dfrac{\pi_{1k}}{\Gamma\{\Box^{k} A\}} \qquad \vdots \quad k<\omega}{\Gamma\{\boxast A\}}
  \qquad
  \diamondast\;\dfrac{\dfrac{\pi_{21}}{\Gamma\{\diamondast \bar{A}, \Diamond^{j} \bar{A}\}}}{\Gamma\{\diamondast \bar{A}\}}
}{\Gamma\{\emptyset\}}\;\mathsf{cut}_{\omega+\sigma}
$$

$(\boxast - \diamondast)$:

$$\text{cut}_{\omega+\sigma}\ \dfrac{\boxast\ \dfrac{\vdots \quad \overset{\pi_{1k}}{\Gamma\{\Box^k A\}} \quad \vdots}{\Gamma\{\boxast A\}}\ {\scriptstyle k<\omega} \qquad \diamondast\ \dfrac{\overset{\pi_{21}}{\Gamma\{\diamondast \bar{A}, \Diamond^j \bar{A}\}}}{\Gamma\{\diamondast \bar{A}\}}}{\Gamma\{\emptyset\}}$$

$\rightsquigarrow$

$$\text{cut}_{\sigma+(j\cdot h)}\ \dfrac{\overset{\pi_{1j}}{\Gamma\{\Box^j A\}} \qquad \text{cut}_{\omega+\sigma}\ \dfrac{\boxast\ \dfrac{\vdots \quad \text{wk}\ \dfrac{\overset{\pi_{1k}}{\Gamma\{\Box^k A\}}}{\Gamma\{\Box^k A, \Diamond^j \bar{A}\}} \quad \vdots}{\Gamma\{\boxast A, \Diamond^j \bar{A}\}}\ {\scriptstyle k<\omega} \qquad \overset{\pi_{21}}{\Gamma\{\diamondast \bar{A}, \Diamond^j \bar{A}\}}}{\Gamma\{\Diamond^j \bar{A}\}}}{\Gamma\{\emptyset\}}$$

First Elimination Lemma

If $\mathsf{D_C} \vdash_{\gamma+1}^{\alpha} \Gamma$ then $\mathsf{D_C} \vdash_{\gamma}^{2^{\alpha}} \Gamma$.

First Elimination Lemma

If $\mathsf{D_C} \overset{\alpha}{\underset{\gamma+1}{\vdash}} \Gamma$ then $\mathsf{D_C} \overset{2^\alpha}{\underset{\gamma}{\vdash}} \Gamma$.

Second Elimination Lemma

If $\mathsf{D_C} \overset{\alpha}{\underset{\beta+\omega^\gamma}{\vdash}} \Gamma$ then $\mathsf{D_C} \overset{\varphi_\gamma \alpha}{\underset{\beta}{\vdash}} \Gamma$.

First Elimination Lemma

If $\mathsf{D_C} \vdash^{\alpha}_{\gamma+1} \Gamma$ then $\mathsf{D_C} \vdash^{2^{\alpha}}_{\gamma} \Gamma$.

Second Elimination Lemma

If $\mathsf{D_C} \vdash^{\alpha}_{\beta+\omega^{\gamma}} \Gamma$ then $\mathsf{D_C} \vdash^{\varphi_{\gamma}\alpha}_{\beta} \Gamma$.

Theorem (Cut Elimination)

If A is a valid formula then $\mathsf{D_C} \vdash^{\varphi_2 0}_{0} A$.

Future work

- Extend the result to Alberucci/Jäger's system
- Extend the result to S5-based common knowledge
- Extend the result to the μ-calculus
- Find lower bounds
- Do syntactic cut elimination in a finitary system