How to Universally Close the Existential-Rule

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Free Logic

	classical predicate logic	
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	classical predicate logic	**free logic**
$\forall x A \supset \exists x A$ is:	valid	not valid
which domains:	the non-empty ones	all of them

Problem

How to get a sequent system for free logic?

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A Sequent System for Classical Predicate Logic

$$\Gamma, a, \bar{a} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{ctr}\,\frac{\Gamma, A, A}{\Gamma, A}$$

$$\forall\,\frac{\Gamma, A[x := y]}{\Gamma, \forall x A}\ \text{y not free in conclusion} \qquad \exists\,\frac{\Gamma, A[x := y]}{\Gamma, \exists x A}$$

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A Solution (Bencivenga, eighties)

$$\forall_{\mathsf{F}}\,\frac{\Gamma, A[x := y], \overline{\mathcal{E}(y)}}{\Gamma, \forall x A}\ \text{y not free in conclusion} \qquad \exists_{\mathsf{F}}\,\frac{\Gamma, A[x := y] \quad \Gamma, \mathcal{E}(y)}{\Gamma, \exists x A}$$

Craig Interpolation

If $A \supset B$ then there is a C in the language $L(A) \cap L(B)$ such that $A \supset C$ and $C \supset B$.

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From Bencivenga's System we only get:

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Problem

Find a sequent system which gives the usual interpolation result.

Hilbert Systems

All propositional tautologies, modus ponens, plus:

Predicate Logic

$A \to \forall x A$ if x not free in A

$\forall x (A \to B) \to (\forall x A \to \forall x B)$

$\forall x A \to A[x := y]$

$$\text{Gen}\ \frac{A}{\forall x A}$$

Hilbert Systems

All propositional tautologies, modus ponens, plus:

Predicate Logic	Free Logic
$A \rightarrow \forall x A$ if x not free in A	$A \leftrightarrow \forall x A$ if x not free in A
$\forall x(A \rightarrow B) \rightarrow (\forall x A \rightarrow \forall x B)$	$\forall x(A \rightarrow B) \rightarrow (\forall x A \rightarrow \forall x B)$
$\forall x A \rightarrow A[x := y]$	$\forall y(\forall x A \rightarrow A[x := y])$
Gen $\dfrac{A}{\forall x A}$	Gen $\dfrac{A}{\forall x A}$
	$\forall x \forall y A \leftrightarrow \forall y \forall x A$

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Nested Sequents

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Example

$$A,\ B \vee C,\ \forall x[A,\ B],\ \forall y[\forall z[\exists x E,\ \forall y F,\ G]]$$

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A context *binds y* if it is of the form $\Gamma_1\{\forall y[\Gamma_2\{\ \}]\}$.

A Nested Sequent System for Predicate Logic

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$$\forall\,\frac{\Gamma\{\forall x[A]\}}{\Gamma\{\forall x A\}} \qquad \mathsf{scp}\,\frac{\forall x[\Gamma\{\Delta\}]}{\Gamma\{\forall x[\Delta]\}}$$ where x does not occur in $\Gamma\{\ \}$

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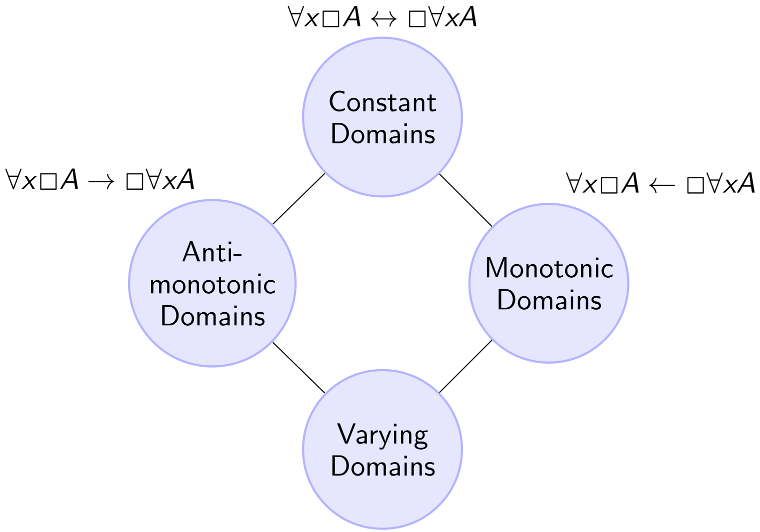
- complete for (sentences of) predicate logic
- subsystem without $\exists_2$-rule complete for free logic
- has syntactic cut-elimination
- usual interpolation follows easily

The General Problem of Quantifiers in Non-Classical Logics

Four Different Modal Predicate Logics



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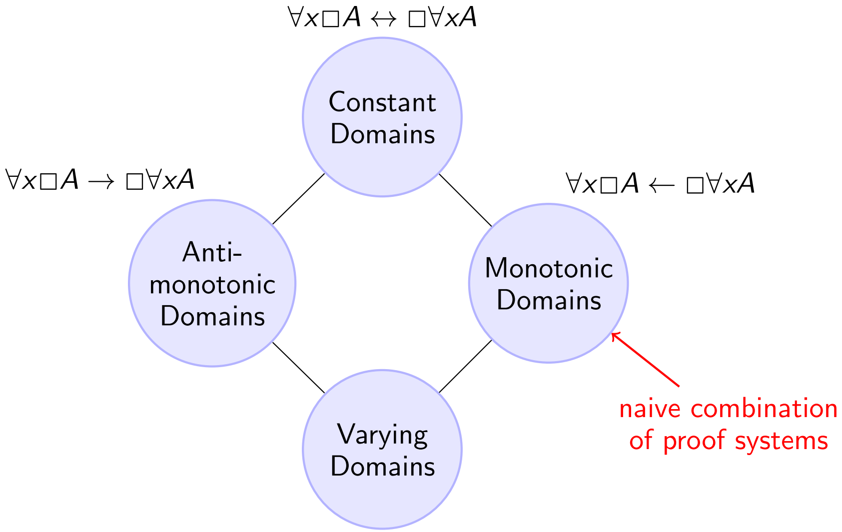

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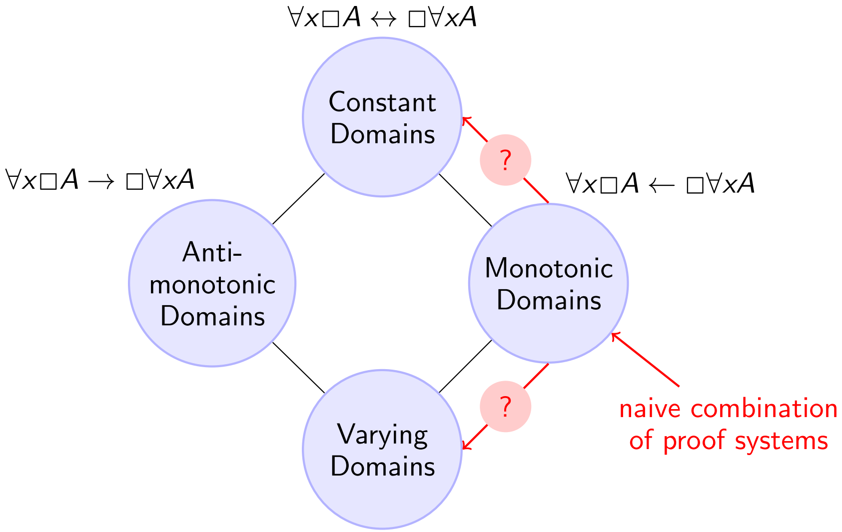

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- Find a systematic cut-free axiomatisation of the four modal predicate logics.

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- Find a system for Gödel-Dummett-Logic with varying domains.
- Find a system for bi-intuitionistic logic with varying domains