On a Mismatch in the Structure of Proofs

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Outline

1. Proof Theory and Why it is Interesting
2. Modal Predicate Logic and its Lack of Proof Systems
3. A Nested Sequent System for Modal Predicate Logic

Outline

Proof Theory

- *Proof theory* is a branch of mathematics which studies proofs as mathematical objects.
- It started out as a way to make sure that a logical system does *not prove contradictions*. Such a system is called *consistent*.

Some History

1879 Frege's Begriffsschrift
(let's formalise mathematics)

1903 Russel's Paradox
(your formalisation is inconsistent)

1920 Hilbert's Program
(let's rule out inconsistencies)

1931 Gödel's Incompleteness Theorems
(you can't)

1935 Gentzen's Sequent Calculus
(let's anyway do what we can)

The Sequent Calculus

$$\Gamma, a, \bar{a} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

The Sequent Calculus

$$\Gamma, a, \bar{a} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\text{cut}\,\frac{\Gamma, A \quad \Gamma, \bar{A}}{\Gamma}$$

The Sequent Calculus

$$\Gamma, a, \bar{a} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\text{cut}\,\frac{\bar{A} \to \Gamma \quad A \to \Gamma}{\Gamma}$$

Example Proof

$$
\vee\,\dfrac{\vee\,\dfrac{\wedge\,\dfrac{a, b, \bar{a} \qquad a, b, \bar{b}}{a, b, \bar{a} \wedge \bar{b}}}{a, b \vee (\bar{a} \wedge \bar{b})}}{a \vee (b \vee (\bar{a} \wedge \bar{b}))}
$$

Cut Elimination

Theorem

If a formula is provable then it is provable without cut.

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Proof

Take a proof with cuts. Push the cuts upwards until they disappear. Now you have a proof without cuts.

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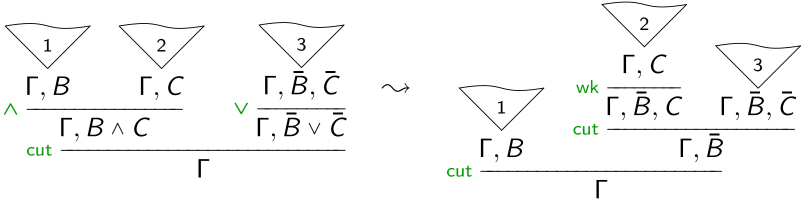

Consequences of Cut Elimination

1. *Properties of the logic*: consistency, decidability, interpolation, Herbrand's Theorem etc.
2. Easily implemented for *automated reasoning*
3. suitable for *proof-search-as-programming* aka logic programming
4. suitable for *proof-normalisation-as programming* aka functional programming

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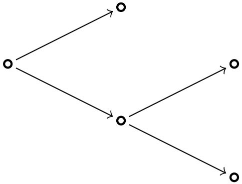
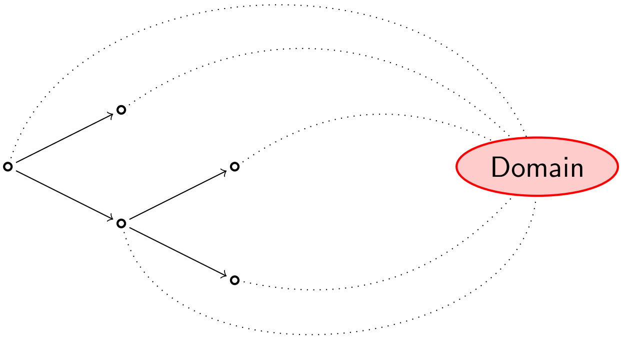
Modalities and Quantifiers

$\Box A, \Diamond A$



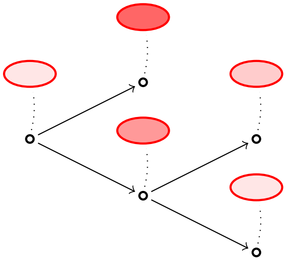
$\exists x A, \forall x A$

Modalities and Quantifiers



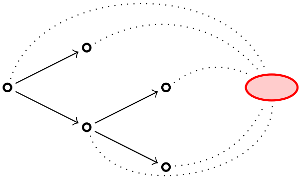
"Constant Domains"

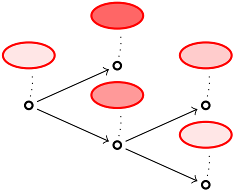
Modalities and Quantifiers



"Varying Domains"

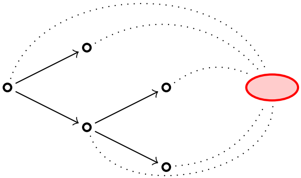
The Barcan Formula

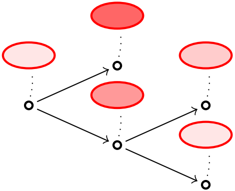




$$\forall x \Box A \leftrightarrow \Box \forall x A$$

The Barcan Formula





valid

$\forall x \Box A \leftrightarrow \Box \forall x A$

not valid
(neither direction)

The Naive Proof System

$$\Gamma, a, \bar{a}$$

$$\wedge\ \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B}$$

$$\vee\ \frac{\Gamma, A, B}{\Gamma, A \vee B}$$

$$\text{ctr}\ \frac{\Gamma, A, A}{\Gamma, A}$$

$$\forall\ \frac{\Gamma, A[x := u]}{\Gamma, \forall x A}\quad \begin{array}{c} u \text{ not free} \\ \text{in conclusion} \end{array}$$

$$\exists\ \frac{\Gamma, A[x := u]}{\Gamma, \exists x A}$$

$$\Box\ \frac{\Gamma, A}{\Diamond\Gamma, \Box A, \Sigma}$$

Neither Varying nor Constant Domains...

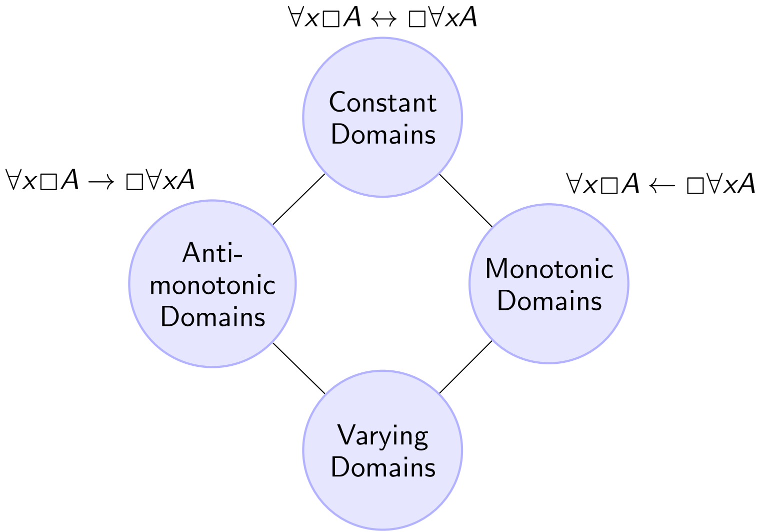
$$
\forall\,\dfrac{\bar{A}[x := u], A[x := z]}{\bar{A}[x := u], \forall x A}
$$
$$
\Box\,\dfrac{\bar{A}[x := u], \forall x A}{\Diamond \bar{A}[x := u], \Box \forall x A}
$$
$$
\exists\,\dfrac{\Diamond \bar{A}[x := u], \Box \forall x A}{\exists x \Diamond \bar{A}, \Box \forall x A}
$$
$$
\vee\,\dfrac{\exists x \Diamond \bar{A}, \Box \forall x A}{\exists x \Diamond \bar{A} \vee \Box \forall x A}
$$
$$
=\,\dfrac{\exists x \Diamond \bar{A} \vee \Box \forall x A}{\forall x \Box A \to \Box \forall x A}
$$

Neither Varying nor Constant Domains...

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bar{A}[x := u], A[x := {\color{red}z}]}{\bar{A}[x := u], \forall x A}\,\forall}{\Diamond\bar{A}[x := u], \Box\forall x A}\,\Box}{\exists x \Diamond \bar{A}, \Box\forall x A}\,\exists}{\exists x \Diamond \bar{A} \vee \Box\forall x A}\,\vee}{\forall x \Box A \to \Box\forall x A}\,=
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A[x := u], \bar{A}[x := u]}{A[x := u], \exists x \bar{A}}\,\exists}{\Box A[x := u], \Diamond\exists x \bar{A}}\,\Box}{\forall x \Box A, \Diamond\exists x \bar{A}}\,\forall}{\forall x \Box A \vee \Diamond\exists x \bar{A}}\,\vee}{\forall x \Box A \leftarrow \Box\forall x A}\,=
$$

More Domain Types



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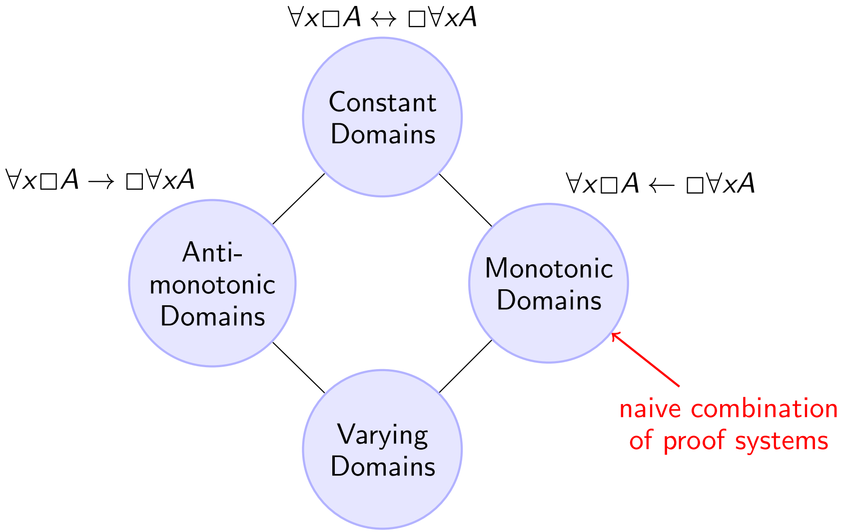

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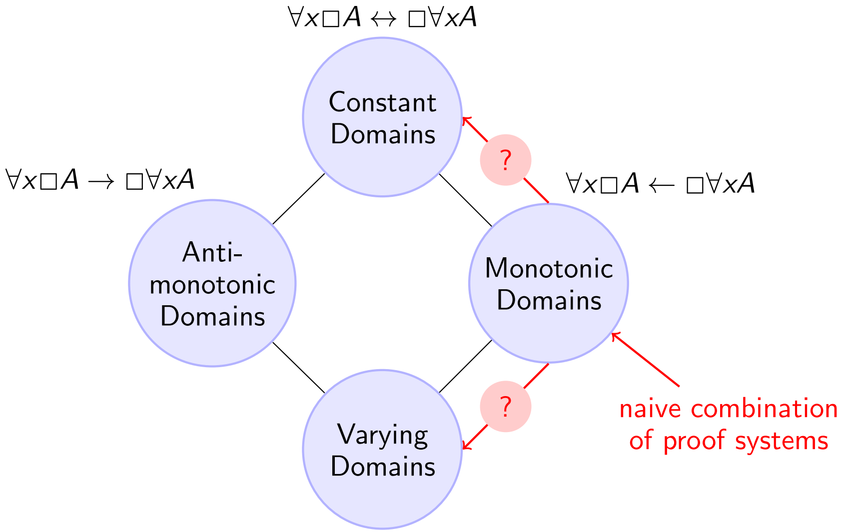

What goes wrong in the naive system?

$$
\begin{array}{ll}
 & \bar{A}[x := u], A[x := z] \\
\forall & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & \bar{A}[x := u], \forall x A \\
\Box & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & \Diamond \bar{A}[x := u], \Box \forall x A \\
\exists & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & \exists x \Diamond \bar{A}, \Box \forall x A \\
\vee & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & \exists x \Diamond \bar{A} \vee \Box \forall x A \\
= & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & \forall x \Box A \to \Box \forall x A
\end{array}
$$

$$
\begin{array}{ll}
 & A[x := u], \bar{A}[x := u] \\
\exists & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & A[x := u], \exists x \bar{A} \\
\Box & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & \Box A[x := u], \Diamond \exists x \bar{A} \\
\forall & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & \forall x \Box A, \Diamond \exists x \bar{A} \\
\vee & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & \forall x \Box A \vee \Diamond \exists x \bar{A} \\
= & \overline{\qquad\qquad\qquad\qquad\qquad} \\
 & \forall x \Box A \leftarrow \Box \forall x A
\end{array}
$$

What goes wrong in the naive system?

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bar{A}[x := u], A[x := {\color{red}z}]}{\bar{A}[x := u], \forall x A}\ \forall}{\Diamond\bar{A}[x := u], \Box\forall x A}\ \Box}{\exists x \Diamond \bar{A}, \Box\forall x A}\ \exists}{\exists x \Diamond \bar{A} \vee \Box\forall x A}\ \vee}{\forall x \Box A \to \Box\forall x A}\ =
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A[x := u], \bar{A}[x := u]}{A[x := u], \exists x \bar{A}}\ \exists}{\Box A[x := u], \Diamond\exists x \bar{A}}\ \Box}{\forall x \Box A, \Diamond\exists x \bar{A}}\ \forall}{\forall x \Box A \vee \Diamond\exists x \bar{A}}\ \vee}{\forall x \Box A \leftarrow \Box\forall x A}\ =
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What goes wrong in the naive system?

$$\forall\ \dfrac{\bar{A}[x := u],\ A[x := z]}{\bar{A}[x := u],\ \forall x A}$$

$$\Box\ \dfrac{\bar{A}[x := u],\ \forall x A}{\Diamond \bar{A}[x := u],\ \Box \forall x A}$$

$$\exists\ \dfrac{\Diamond \bar{A}[x := u],\ \Box \forall x A}{\exists x \Diamond \bar{A},\ \Box \forall x A}$$

$$\vee\ \dfrac{\exists x \Diamond \bar{A},\ \Box \forall x A}{\exists x \Diamond \bar{A} \vee \Box \forall x A}$$

$$=\ \dfrac{\exists x \Diamond \bar{A} \vee \Box \forall x A}{\forall x \Box A \to \Box \forall x A}$$

$$\exists\ \dfrac{A[x := u],\ \bar{A}[x := u]}{A[x := u],\ \exists x \bar{A}}$$

$$\Box\ \dfrac{A[x := u],\ \exists x \bar{A}}{\Box A[x := u],\ \Diamond \exists x \bar{A}}$$

$$\forall\ \dfrac{\Box A[x := u],\ \Diamond \exists x \bar{A}}{\forall x \Box A,\ \Diamond \exists x \bar{A}}$$

$$\vee\ \dfrac{\forall x \Box A,\ \Diamond \exists x \bar{A}}{\forall x \Box A \vee \Diamond \exists x \bar{A}}$$

$$=\ \dfrac{\forall x \Box A \vee \Diamond \exists x \bar{A}}{\forall x \Box A \leftarrow \Box \forall x A}$$

What goes wrong in the naive system?

$$
=\dfrac{\vee\,\dfrac{\exists\,\dfrac{\Box\,\dfrac{\forall\,\dfrac{\bar{A}[x := u],\ A[x := z]}{\bar{A}[x := u],\ \forall x A}}{\Diamond\bar{A}[x := u],\ \Box\forall x A}}{\exists x\Diamond\bar{A},\ \Box\forall x A}}{\exists x\Diamond\bar{A} \vee \Box\forall x A}}{\forall x\Box A \to \Box\forall x A}
$$

$$
=\dfrac{\vee\,\dfrac{\forall\,\dfrac{\Box\,\dfrac{\exists\,\dfrac{A[x := u],\ \bar{A}[x := u]}{A[x := u],\ \exists x\bar{A}}}{\Box A[x := u],\ \Diamond\exists x\bar{A}}}{\forall x\Box A,\ \Diamond\exists x\bar{A}}}{\forall x\Box A \vee \Diamond\exists x\bar{A}}}{\forall x\Box A \leftarrow \Box\forall x A}
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What goes wrong in the naive system?

$$
\textstyle = \dfrac{\vee \dfrac{\exists \dfrac{\Box \dfrac{\forall \dfrac{\bar{A}[x := u],\ A[x := z]}{\bar{A}[x := u],\ \forall x A}}{\Diamond \bar{A}[x := u],\ \Box \forall x A}}{\exists x \Diamond \bar{A},\ \Box \forall x A}}{\exists x \Diamond \bar{A} \vee \Box \forall x A}}{\forall x \Box A \to \Box \forall x A}
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$$
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$$

It's not so clear how to solve our problem in the sequent system.
Let's look at a Hilbert system, where it is easily solved.

A Hilbert System

All propositional tautologies, modus ponens, plus:

$\Box(A \to B) \to (\Box A \to \Box B)$

$$\text{Nec}\ \frac{A}{\Box A}$$

$A \to \forall x A \qquad$ if x not free in A

$\forall x(A \to B) \to (\forall x A \to \forall x B)$

$\forall x A \to A[x := y]$

$$\text{Gen}\ \frac{A}{\forall x A}$$

A Hilbert System

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Nec $\dfrac{A}{\Box A}$

$A \to \forall x A$ if x not free in A

$\forall x(A \to B) \to (\forall x A \to \forall x B)$

$\forall x A \to A[x := y]$ not valid for varying domains!

Gen $\dfrac{A}{\forall x A}$

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$A \to \forall x A \qquad$ if x not free in A

$\forall x(A \to B) \to (\forall x A \to \forall x B)$

$\forall y(\forall x A \to A[x := y])$ valid for varying domains

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A Hilbert System

All propositional tautologies, modus ponens, plus:

$\Box(A \to B) \to (\Box A \to \Box B)$

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$A \leftrightarrow \forall x A$ if x not free in A

$\forall x(A \to B) \to (\forall x A \to \forall x B)$

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$\forall x \forall y A \leftrightarrow \forall y \forall x A$

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The Problem

Problem

Find a systematic cut-free axiomatisation of the four modal predicate logics.

Subproblem

Find a cut-free system for predicate logic, which does not prove $\forall x A \rightarrow A[x := y]$.

Question

Can we get it in the same way we just got the Hilbert-system?

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How to weaken this system?

$$\Gamma, a, \bar{a} \qquad \wedge\,\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \vee\,\frac{\Gamma, A, B}{\Gamma, A \vee B} \qquad \text{ctr}\,\frac{\Gamma, A, A}{\Gamma, A}$$

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Nested Sequents

Solution

Extend sequents by structural connective for $\forall x$, denoted $\forall x[\ ]$.

Idea

$\forall x[\ ]$ is for $\forall x$ what "," is for $\vee$.

Example

$$A, B \vee C, \forall x[A, B], \forall y[\forall z[\exists x E, \forall y F, G]]$$

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A Nested Sequent System

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$$\forall\,\frac{\Gamma\{\forall x[A]\}}{\Gamma\{\forall x A\}} \qquad \mathsf{scp}\,\frac{\forall x[\Gamma\{\Delta\}]}{\Gamma\{\forall x[\Delta]\}}$$ where x does not occur in $\Gamma\{\ \}$

$$\exists_1\,\frac{\Gamma\{A[x := y]\}}{\Gamma\{\exists x A\}}$$ where $\Gamma\{\}$ binds y

$$\exists_2\,\frac{\Gamma\{\forall x[A]\}}{\Gamma\{\exists x A\}}$$

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$$\exists_1 \frac{\Gamma\{A[x := y]\}}{\Gamma\{\exists x A\}} \text{ where } \Gamma\{\} \text{ binds } y \qquad \exists_2 \frac{\Gamma\{\forall x[A]\}}{\Gamma\{\exists x A\}}$$

A Nested Sequent System for Predicate Logic

- sound for varying domains
  $\nvdash \forall x A \rightarrow A[x := y]$
- complete for sentences of predicate logic
  $\vdash \forall y(\forall x A \rightarrow A[x := y])$
- has syntactic cut-elimination
- subsystem without $\exists_2$ captures possibly-empty domains
- has the "free variable property"

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A Nested Sequent System for Modal Predicate Logic

Varying Domains

$$\text{scp}\ \frac{\Gamma\{\forall x[\Lambda\{\Delta\}]\}}{\Gamma\{\Lambda\{\forall x[\Delta]\}\}}$$ where x does not occur in the local context $\Lambda\{\}$

$$\exists_1\ \frac{\Gamma\{A[x := y]\}}{\Gamma\{\exists x A\}}$$ where $\Gamma\{\}$ locally binds y

$$\Box\ \frac{\Gamma\{\Box[A]\}}{\Gamma\{\Box A\}} \qquad \Diamond\ \frac{\Gamma\{\Box[A, \Delta]\}}{\Gamma\{\Diamond A, \Box[\Delta]\}}$$

Barcan and Converse Barcan Rule

$$\text{ba}\ \frac{\Gamma\{\forall x[\Lambda\{\Delta\}]\}}{\Gamma\{\Lambda\{\forall x[\Delta]\}\}}$$ where x does not occur in $\Lambda\{\}$

$$\text{cba}\ \frac{\Gamma\{A[x := y]\}}{\Gamma\{\exists x A\}}$$ where $\Gamma\{\}$ binds y